Time discretization and quantization methods for optimal multiple switching problem*

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Abstract

In this paper, we study probabilistic numerical methods based on optimal quantization algorithms for computing the solution to optimal multiple switching problems with regime-dependent state process. We first consider a discrete-time approximation of the optimal switching problem, and analyze its rate of convergence. Given a time step h, the error is in general of order $(h\log(1/h))^{1/2}$, and of order $h^{1/2}$ when the switching costs do not depend on the state process. We next propose quantization numerical schemes for the space discretization of the discrete-time Euler state process. A Markovian quantization approach relying on the optimal quantization of the normal distribution arising in the Euler scheme is analyzed. In the particular case of uncontrolled state process, we describe an alternative marginal quantization method, which extends the recursive algorithm for optimal stopping problems as in [2]. A priori L^p-error estimates are stated in terms of quantization errors. Finally, some numerical tests are performed for an optimal switching problem with two regimes.

Key words: Optimal switching, quantization of random variables, discrete-time approximation, Markov chains, numerical probability.

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1 Introduction

On some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let us introduce the controlled regime-switching diffusion in $\mathbb{R}^d$ governed by

$$dX_t \;=\; b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t,$$

where W is a standard d-dimensional Brownian motion, $\alpha = (\tau_n, \iota_n)_n \in \mathcal{A}$ is the switching control represented by a nondecreasing sequence of stopping times (τ_n) together with a sequence (ι_n) of $\mathcal{F}_{\tau_n}$-measurable random variables valued in a finite set $\{1, \ldots, q\}$, and α_t is the current regime process, i.e. $\alpha_t = \iota_n$ for $\tau_n \leq t < \tau_{n+1}$. We then consider the optimal switching problem over a finite horizon:

$$V_0 \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\int_0^T f(X_t, \alpha_t)dt + g(X_T, \alpha_T) - \sum_{\tau_n \leq T} c(X_{\tau_n}, \iota_{n-1}, \iota_n)\Big]. \tag{1.1}$$

Optimal switching problems can be seen as sequential optimal stopping problems belonging to the class of impulse control problems, and arise in many applied fields, for example in real option pricing in economics and finance. It has attracted a lot of interest during the past decades, and we refer to Chapter 5 in the book [17] and the references therein for a survey of some applications and results in this topic. It is well-known that optimal switching problems are related via the dynamic programming approach to a system of variational inequalities with inter-connected obstacles in the form:

$$\begin{aligned} \min\Big[-\frac{\partial v_i}{\partial t} - b(x,i).D_x v_i - \frac{1}{2}\mathrm{tr}(\sigma(x,i)\sigma(x,i)' D_x^2 v_i) - f(x,i)\;, \\ v_i - \max_{j\neq i}(v_j - c(x,i,j))\Big] \;=\; 0 \quad \text{on } [0,T)\times\mathbb{R}^d, \end{aligned} \tag{1.2}$$

together with the terminal condition $v_i(T,x) = g(x,i)$, for any $i = 1, \ldots, q$. Here $v_i(t,x)$ is the value function to the optimal switching problem starting at time $t \in [0,T]$ from the state $X_t = x \in \mathbb{R}^d$ and the regime $\alpha_t = i \in \{1, \ldots, q\}$, and the solution to the system (1.2) has to be understood in the weak sense, e.g. viscosity sense.

The purpose of this paper is to solve numerically the optimal switching problem (1.1), and consequently the system of variational inequalities (1.2). These equations can be solved by analytical methods (finite differences, finite elements, etc ...), see e.g. [14], but are known to require heavy computations, especially in high dimension. Alternatively, when the state process is uncontrolled, i.e. regime-independent, optimal switching problems are connected to multi-dimensional reflected Backward Stochastic Differential Equations (BSDEs) with oblique reflections, as shown in [9] and [10], and the recent paper [5] introduced a discretely obliquely reflected numerical scheme to solve such BSDEs. From a computational viewpoint, there are rather few papers dealing with numerical experiments for optimal switching problems. The special case of two regimes for switching problems can be reduced to the resolution of a single BSDE with two reflecting barriers when considering the difference value process, and is exploited numerically in [8]. We mention also the paper [4], which solves an optimal switching problem with three regimes by considering a cascade of reflected BSDEs with one reflecting barrier derived from an iteration on the number of switches.

We propose probabilistic numerical methods based on dynamic programming and optimal quantization methods combined with a suitable time discretization procedure for computing the solution to optimal multiple switching problem. Quantization methods were introduced in [2] for solving variational inequality with given obstacle associated to optimal stopping problem of some diffusion process (X_t). The basic idea is the following. One first approximates the (continuous-time) optimal stopping problem by the Snell envelope for the Markov chain $(\bar{X}_{t_k})$ defined as the Euler scheme of the (uncontrolled) diffusion X, and then spatially discretize each random vector $\bar{X}_{t_k}$ by a random vector taking finite values through a quantization procedure. More precisely, $(\bar{X}_{t_k})_k$ is approximated by $(\hat{X}_k)_k$ where $\hat{X}_k$ is the projection of $\bar{X}_{t_k}$ on a finite grid in the state space following the closest neighbor rule. The induced L^p-quantization error, $\|\bar{X}_{t_k} - \hat{X}_k\|_p$, depends only on the distribution of $\bar{X}_{t_k}$ and the grid, which may be chosen in order to minimize the quantization error. Such an optimal choice, called optimal quantization, is achieved by the competitive learning vector quantization algorithm (or Kohonen algorithm) developed in full details in [2]. One finally computes the approximation of the optimal stopping problem by a quantization tree algorithm, which mimics the backward dynamic programming of the Snell envelope. In this paper, we develop quantization methods to our general framework of optimal switching problem. With respect to standard optimal stopping problems, some new features arise on one hand from the regime-dependent state process, and on the other hand from the multiple switching times, and the discrete sum for the cumulated switching costs.

We first study a time discretization of the optimal switching problem by considering an Euler-type scheme with step $h = T/m$ for the regime-dependent state process (X_t) controlled by the switching strategy α:

$$\bar{X}_{t_{k+1}} \quad = \quad \bar{X}_{t_k} + b(\bar{X}_{t_k}, \alpha_{t_k})h + \sigma(\bar{X}_{t_k}, \alpha_{t_k})\sqrt{h}\, \vartheta_{k+1}, \;\; t_k = kh, \; k = 0, \ldots, m, \tag{1.3}$$

where ϑ_k, $k = 1, \ldots, m$, are iid, and $\mathcal{N}(0, I_d)$-distributed. We then introduce the optimal switching problem for the discrete-time process $(\bar{X}_{t_k})$ controlled by switching strategies with stopping times valued in the discrete time grid $\{t_k, k = 0, \ldots, m\}$. The convergence of this discrete-time problem is analyzed, and we prove that the error is in general of order $(h \log(1/h))^{\frac{1}{2}}$, and of order $h^{\frac{1}{2}}$,as for optimal stopping problems, when the switching costs $c(x, i, j) \equiv c(i, j)$ do not depend on the state process. Arguments of the proof rely on a regularity result of the controlled diffusion with respect to the switching strategy, and moment estimates on the number of switches. This improves and extends the convergence rate result in [5] derived in the case where X is regime-independent.

Next, we propose approximation schemes by quantization for computing explicitly the solution to the discrete-time optimal switching problem. Since the controlled Markov chain $(\bar{X}_{t_k})_k$ cannot be directly quantized as in standard optimal stopping problems, we adopt a Markovian quantization approach in the spirit of [15], by considering an optimal quantization of the Gaussian random vector ϑ_{k+1} arising in the Euler scheme (1.3). A quantization tree algorithm is then designed for computing the approximating value function, and we provide error estimates in terms of the quantization errors $\|\vartheta_k - \hat{\vartheta}_k\|_p$ and state space grid parameters. Alternatively, in the case of regime-independent state process, we propose a quantization algorithm in the vein of [2] based on marginal quantization of the uncontrolled

Markov chain $(\bar{X}_{t_k})_k$. A priori L^p-error estimates are also established in terms of quantization errors $\|\bar{X}_{t_k} - \hat{X}_k\|_p$. Finally, some numerical tests on the two quantization algorithms are performed for an optimal switching problem with two regimes.

The plan of this paper is organized as follows. Section 2 formulates the optimal switching problem and sets the standing assumptions. We also show some preliminary results about moment estimates on the number of switches. We describe in Section 3 the time discretization procedure, and study the rate of convergence of the discrete-time approximation for the optimal switching problem. Section 4 is devoted to the approximation schemes by quantization for the explicit computation of the value function to the discrete-time optimal switching problem, and to the error analysis. Finally, we illustrate our results with some numerical tests in Section 5.

2 Optimal switching problem

2.1 Formulation and assumptions

We formulate the finite horizon multiple switching problem. Let us fix a finite time $T \in (0,\infty)$, and some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Let $\mathbb{I}_q = \{1, \ldots, q\}$ be the set of all possible regimes (or activity modes). A switching control is a double sequence $\alpha = (\tau_n, \iota_n)_{n\geq 0}$, where (τ_n) is a nondecreasing sequence of stopping times, and ι_n are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $\mathbb{I}_q$. The switching control $\alpha = (\tau_n, \iota_n)$ is said to be admissible, and denoted by $\alpha \in \mathcal{A}$, if there exists an integer-valued random variable N with $\tau_N > T$ a.s. Given $\alpha = (\tau_n, \iota_n)_{n\geq 0} \in \mathcal{A}$, we may then associate the indicator of the regime value defined at any time $t \in [0,T]$ by

$$I_t = \iota_0 \mathbf{1}_{\{0\leq t<\tau_0\}} + \sum_{n\geq 0} \iota_n \mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}},$$

which we shall sometimes identify with the switching control α, and we introduce $N(\alpha)$ the (random) number of switches before T:

$$N(\alpha) = \#\big\{n \geq 1 \ : \tau_n \leq T\big\}\ .$$

For $\alpha \in \mathcal{A}$, we consider the controlled regime-switching diffusion process valued in $\mathbb{R}^d$, governed by the dynamics

$$dX_s = b(X_s, I_s)ds + \sigma(X_s, I_s)dW_s, \quad X_0 = x_0 \in \mathbb{R}^d, \tag{2.1}$$

where W is a standard d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. We shall assume that the coefficients $b_i = b(.,i)$: $\mathbb{R}^d \to \mathbb{R}^d$, and $\sigma_i(.) = \sigma(.,i) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $i \in \mathbb{I}_q$, satisfy the usual Lipschitz conditions.

We are given a running reward, terminal gain functions $f, g : \mathbb{R}^d \times \mathbb{I}_q \to \mathbb{R}$, and a cost function $c : \mathbb{R}^d \times \mathbb{I}_q \times \mathbb{I}_q \to \mathbb{R}$, and we set $f_i(.) = f(.,i)$, $g_i(.) = g(.,i)$, $c_{ij}(.) = c(.,i,j)$, $i,j \in \mathbb{I}_q$. We shall assume the Lipschitz condition:

(Hl) The coefficients f_i, g_i and c_{ij}, $i,j \in \mathbb{I}_q$ are Lipschitz continuous on $\mathbb{R}^d$.

We also make the natural triangular condition on the functions c_{ij} representing the instantaneous cost for switching from regime i to j:

(Hc)

$$
\begin{aligned}
c_{ii}(.) &= 0, \quad i \in \mathbb{I}_q, \\
\inf_{x\in\mathbb{R}^d} c_{ij}(x) &> 0, \quad \text{for } i,j \in \mathbb{I}_q,\ j \neq i, \\
\inf_{x\in\mathbb{R}^d} \big[c_{ij}(x) + c_{jk}(x) - c_{ik}(x)\big] &> 0, \quad \text{for } i,j,k \in \mathbb{I}_q,\ j \neq i,k.
\end{aligned}
$$

The triangular condition on the switching costs c_{ij} in **(Hc)** means that when one changes from regime i to some regime j, then it is not optimal to switch again immediately to another regime, since it would induce a higher total cost, and so one should stay for a while in the regime j.

The expected total profit over $[0,T]$ for running the system with the admissible switching control $\alpha = (\tau_n, \iota_n) \in \mathcal{A}$ is given by:

$$
J_0(\alpha) = \mathbb{E}\Big[\int_0^T f(X_t, I_t)dt + g(X_T, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}, \iota_{n-1}, \iota_n)\Big].
$$

The maximal profit is then defined by

$$
V_0 = \sup_{\alpha\in\mathcal{A}} J_0(\alpha). \tag{2.2}
$$

The dynamic version of this optimal switching problem is formulated as follows. For (t,i) $\in [0,T]\times\mathbb{I}_q$, we denote by $\mathcal{A}_{t,i}$ the set of admissible switching controls $\alpha = (\tau_n, \iota_n)$ starting from i at time t, i.e. $\tau_0 = t$, $\iota_0 = i$. Given $\alpha \in \mathcal{A}_{t,i}$, and $x \in \mathbb{R}^d$, and under the Lipschitz conditions on b, σ, there exists a unique strong solution to (2.1) starting from x at time t, and denoted by $\{X_s^{t,x,\alpha}, t \leq s \leq T\}$. It is then given by

$$
X_s^{t,x,\alpha} = x + \sum_{\tau_n\leq s}\int_{\tau_n}^{\tau_{n+1}\wedge s} b_{\iota_n}(X_u^{t,x,\alpha})du + \int_{\tau_n}^{\tau_{n+1}\wedge s}\sigma_{\iota_n}(X_u^{t,x,\alpha})dW_u,\ t \leq s \leq T. \tag{2.3}
$$

The value function of the optimal switching problem is defined by

$$
v_i(t,x) = \sup_{\alpha\in\mathcal{A}_{t,i}} \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s)ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big], \tag{2.4}
$$

for any $(t,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_q$, so that $V_0 = \max_{i\in\mathbb{I}_q} v_i(0,x_0)$.

For simplicity, we shall also make the assumption

$$
g_i(x) \geq \max_{j\in\mathbb{I}_q}[g_j(x) - c_{ij}(x)], \quad \forall (x,i) \in \mathbb{R}^d\times\mathbb{I}_q. \tag{2.5}
$$

This means that any switching decision at horizon T induces a terminal profit, which is smaller than a no-decision at this time, and is thus suboptimal. Therefore, the terminal condition for the value function is given by:

$$
v_i(T,x) = g_i(x), \quad (x,i) \in \mathbb{R}^d\times\mathbb{I}_q.
$$

Otherwise, it is given in general by $v_i(T,x) = \max_{j\in\mathbb{I}_q}[g_j(x) - c_{ij}(x)]$.

Notations. $|.|$ will denote the canonical Euclidian norm on $\mathbb{R}^d$, and $(.|.)$ the corresponding inner product. For any $p \geq 1$, and Y random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\|Y\|_p = (\mathbb{E}|Y|^p)^{\frac{1}{p}}$.

2.2 Preliminaries

We first show that one can restrict the optimal switching problem to controls α with bounded moments of $N(\alpha)$. More precisely, let us associate to a strategy $\alpha \in \mathcal{A}_{t,i}$, the cumulated cost process $C^{t,x,\alpha}$ defined by

$$C_u^{t,x,\alpha} \;=\; \sum_{n\geq 1} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\mathbf{1}_{\tau_n \leq u}, \qquad t \leq u \leq T.$$

We then consider for $x \in \mathbb{R}^d$ and $K > 0$ the subset $\mathcal{A}_{t,i}^K(x)$ of $\mathcal{A}_{t,i}$ defined by

$$\mathcal{A}_{t,i}^K(x) \;=\; \Big\{\alpha \in \mathcal{A}_{t,i} : \mathbb{E}\big|C_T^{t,x,\alpha}\big|^2 \;\leq\; K(1+|x|^2)\Big\}.$$

Proposition 2.1 *Assume that* **(Hl)** *and* **(Hc)** *holds. Then, there exists some positive constant K s.t.*

$$v_i(t,x) \;=\; \sup_{\alpha\in\mathcal{A}_{t,i}^K(x)} \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s)ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big] \tag{2.6}$$

for any $(t,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_q$.

Remark 2.1 Under the uniformly strict positive condition on the switching costs in **(Hc)**, there exists some positive constant $\eta > 0$ s.t. $N(\alpha) \leq \eta C_T^{t,x,\alpha}$ for any $(t,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_q$, $\alpha \in \mathcal{A}_{t,i}$. Thus, for any $\alpha \in \mathcal{A}_{t,i}^K(x)$, we have

$$\mathbb{E}\big|N(\alpha)\big|^2 \;\leq\; \eta K(1+|x|^2),$$

which means that in the value functions $v_i(t,x)$ of optimal switching problems, one can restrict to controls α for which the second moment of $N(\alpha)$ is bounded by a constant depending on x.

Before proving Proposition 2.1, we need the following Lemmata.

Lemma 2.1 *For all $p \geq 1$, there exists a positive constant K_p such that*

$$\sup_{\alpha\in\mathcal{A}_{t,i}} \Big\|\sup_{s\in[t,T]} \big|X_s^{t,x,\alpha}\big|\Big\|_p \;\leq\; K_p(1+|x|)\,,$$

for all $(t,x,i) \in [0,T]\times\mathbb{R}^d\times\mathbb{I}_q$.

Proof. Fix $p \geq 1$. Then, we have from the definition of $X_s^{t,x,\alpha}$ in(2.3), for $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}_{t,i}$:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[t,r]} \big|X_s^{t,x,\alpha}\big|^p\Big] &\leq K_p\Big(|x|^p + \mathbb{E}\Big[\sum_{\tau_n \leq r} \int_{\tau_n}^{\tau_{n+1}\wedge r} \big|b_{\iota_n}(X_u^{t,x,\alpha})\big|^p du\Big] \\
&\quad + \mathbb{E}\Big[\sup_{s\in[t,r]} \Big|\sum_{\tau_n \leq s} \int_{\tau_n}^{\tau_{n+1}\wedge s} \sigma_{\iota_n}(X_u^{t,x,\alpha}) dW_u\Big|^p\Big]\Big) ,
\end{aligned}$$

for all $r \in [t,T]$. From the linear growth conditions on b_i and σ_i, for $i \in \mathbb{I}_q$, and Burkholder-Davis-Gundy's (BDG) inequality, we then get by Hölder inequality when $p \geq 2$:

$$\mathbb{E}\Big[\sup_{s\in[t,r]} \big|X_s^{t,x,\alpha}\big|^p\Big] \leq K_p\Big(1 + |x|^p + \int_t^r \mathbb{E}\Big[\sup_{s\in[t,u]} \big|X_s^{t,x,\alpha}\big|^p du\Big]\Big) ,$$

for all $r \in [t,T]$. By applying Gronwall's Lemma, we obtain the required estimate for $p \geq 2$, and then also for $p \geq 1$ by Hölder inequality. □

Lemma 2.2 *Under* **(Hl)** *and* **(Hc)***, the functions v_i, $i \in \mathbb{I}_q$, satisfy a linear growth condition, i.e. there exists a constant K such that*

$$|v_i(t,x)| \leq K\big(1 + |x|\big) ,$$

for all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$.

Proof. Under the linear growth condition on f_i, g_i in **(Hl)**, and the nonnegativity of the switching costs in **(Hc)**, there exists some positive constant K s.t.

$$\begin{aligned}
&\mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s) ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big] \\
\leq\ & K\Big(1 + \mathbb{E}\Big[\sup_{u\in[0,T]} \big|X_u^{t,x,\alpha}\big|\Big]\Big),
\end{aligned}$$

for all $(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_q$, $\alpha \in \mathcal{A}_t, i$. By combining with the estimate in Lemma 2.1, this shows that

$$v_i(t,x) \leq K(1 + |x|) .$$

Moreover, by considering the strategy α^0 with no intervention i.e. $N(\alpha^0) = 0$, we have

$$\begin{aligned}
v_i(t,x) &\geq \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha^0}, i) ds + g(X_T^{t,x,\alpha^0}, i)\Big] \\
&\geq -K\Big(1 + \mathbb{E}\Big[\sup_{u\in[0,T]} \big|X_u^{t,x,\alpha}\big|\Big]\Big).
\end{aligned}$$

Again, by the estimate in Lemma 2.1, this proves that

$$v_i(t,x) \geq -K(1 + |x|) ,$$

and therefore the required linear growth condition on v_i. $\square$

We now turn to the proof of the Proposition.

Proof of Proposition 2.1. The proof is done in 4 steps. Given $\alpha \in \mathcal{A}_{t,i}$, we will denote

$$J(t,x,i;\alpha) \;=\; \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, I_s)ds + g(X_T^{t,x,\alpha}, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t,x,\alpha}, \iota_{n-1}, \iota_n)\Big].$$

• *Step 1.* First, we notice that the supremum in the definition of $v_i(t,x)$ may be taken over $\mathcal{A}_{t,i}^s$, where

$$\mathcal{A}_{t,i}^s \;=\; \Big\{\alpha = (\tau_n, \iota_n) \in \mathcal{A}_{t,i} : (\tau_n) \text{ is strictly increasing } \Big\}.$$

Indeed, it is always suboptimal to switch several times at a single date due to the triangular condition **(Hc)**.

• *Step 2.* We now prove that it is enough to take the supremum over the strategies in $\mathcal{A}_{t,i}^{s,\infty}$, where

$$\mathcal{A}_{t,i}^{s,\infty} \;=\; \Big\{\alpha \in \mathcal{A}_{t,i}^s : \mathbb{E}\big|C_T^{t,x,\alpha}\big|^2 \;<\; +\infty\Big\}.$$

For any $\alpha = (\tau_k, \iota_k)_{k\geq 0} \in \mathcal{A}_{t,i}^s$, define $\alpha^n = (\tau_k^n, \iota_k^n)_{k\geq 0}$ as the strategy obtained from α by only keeping the first n switches, i.e.

$$\begin{aligned}(\tau_k^n, \iota_k^n) &= (\tau_k, \iota_k), \quad k \leq n,\\ \tau_k^n &= \infty, \qquad k > n\end{aligned}$$

Note that for each n, $\alpha^n \in \mathcal{A}_{t,i}^{s,\infty}$. Now since α and α^n (and the associated processes) coincide on $\{N(\alpha) \leq n\}$, and by positivity of the switching costs,

$$\begin{aligned}& J(t,x,i;\alpha) - J(t,x,i;\alpha^n)\\ \leq\;& \mathbb{E}\Big[\Big(\int_t^T (f(X_s^{t,x,\alpha}, I_s) - f(X_s^{t,x,\alpha^n}, I_s))ds + g(X_T^{t,x,\alpha}, I_T) - g(X_T^{t,x,\alpha^n}, I_T)\Big)\mathbf{1}_{\{N(\alpha)>n\}}\Big]\\ \leq\;& K(1+|x|)\mathbb{P}\big(N(\alpha) > n\big)^{1/2},\end{aligned}$$

by Cauchy-Schwarz inequality, linear growth of f, g and Lemma 2.1. Hence letting $n \to \infty$, and since $N(\alpha) < \infty$ a.s., we obtain

$$J(t,x,i;\alpha) \leq \liminf_{n\to\infty} J(t,x,i;\alpha^n),$$

which proves the required assertion.

• *Step 3.* To each $\alpha \in \mathcal{A}_{t,i}^{s,\infty}$, we associate the process $(Y^{t,x,\alpha}, Z^{t,x,\alpha})$ solution to the following Backward Stochastic Differential Equation (BSDE)

$$\begin{aligned}Y_u^{t,x,\alpha} \;=\;& g(X_T^{t,x,\alpha}, I_T^\alpha) + \int_u^T f(X_s^{t,x,\alpha}, I_s^\alpha)ds \\ & - \int_u^T Z_s^{t,x,\alpha}dW_s - C_T^{t,x,\alpha} + C_u^{t,x,\alpha}\,, \qquad t \leq u \leq T\end{aligned} \tag{2.7}$$

and satisfying the condition

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|Y_s^{t,x,\alpha}|^2\Big]+\mathbb{E}\Big[\int_t^T|Z_s^{t,x,\alpha}|^2ds\Big] \;<\; \infty.$$

Such a solution exists under **(Hl)**, Lemma 2.1 and $\mathbb{E}\big[|C_T^{t,x,\alpha}|^2\big]<\infty$. Note that taking the expectation in (2.7), $Y_t^{t,x,\alpha}=J(t,x,i;\alpha)$.

We now define for $\tilde{K}>0$,

$$\tilde{\mathcal{A}}_{t,i}^{s,\tilde{K}}(x) \;=\; \Big\{\alpha\in\mathcal{A}_{t,i}^{s,\infty}:\mathbb{E}\Big[\sup_{s\in[t,T]}\big|Y_s^{t,x,\alpha}\big|^2\Big] \;\le\; \tilde{K}(1+|x|^2)\Big\},$$

and claim that for some constant $\tilde{K}$, the supremum in $v_i(t,x)$ may be taken over $\alpha\in\tilde{\mathcal{A}}_{t,i}^{s,\tilde{K}}(x)$. First taking the conditional expectation in (2.7), we have

$$Y_u^{t,x,\alpha}\le v_{I_t}(X_u^{t,x,\alpha},I_u^{\alpha})\le K(1+\big|X_s^{t,x,\alpha}\big|),\quad t\le u\le T,$$

so that by Lemma 2.1 the only restriction is to have a lower bound on $Y_u^{t,x,\alpha}$. As in Lemma 2.2, this is done by considering strategies with fewer interventions. Given $\alpha\in\mathcal{A}_{t,i}^{s,\infty}$, consider the stopping time

$$\tau=\inf\{s\ge t: J(s,X_s^{t,x,\alpha},I_s^{\alpha};\alpha^0)\ge Y_s^{t,x,\alpha}\}$$

where α^0 is the strategy with no switches, and define $\tilde{\alpha}=(\tilde{\tau}_n,\iota_n)$, where

$$\tilde{\tau}_n \;=\; \tau_n\mathbf{1}_{\{\tau_n\le\tau\}}+\infty\mathbf{1}_{\{\tau_n>\tau\}}.$$

Now for each $t\le u\le T$, taking the conditional expectation in (2.7) we obtain

$$\begin{aligned}
&\mathbf{1}_{\{u\le\tau\}}(Y_u^{t,x,\tilde{\alpha}}-Y_u^{t,x,\alpha})\\
=\;&\mathbb{E}\Big[\mathbf{1}_{\{u\le\tau<T\}}\Big(\int_\tau^T f(X_s^{t,x,\tilde{\alpha}},I_s^{\tilde{\alpha}})ds+g(X_T^{t,x,\tilde{\alpha}},I_T^{\tilde{\alpha}})\\
&\quad-\int_\tau^T f(X_s^{t,x,\alpha},I_s)ds-g(X_T^{t,x,\alpha},I_T)+C_T^{t,x,\alpha}-C_\tau^{t,x,\alpha}\Big)\big|\mathcal{F}_u\Big]\\
=\;&\mathbb{E}\Big[\mathbf{1}_{\{u\le\tau<T\}}\big(J(\tau,X_\tau^{t,x,\alpha},I_\tau^{\alpha};\alpha^0)-Y_\tau^{t,x,\alpha}\big)\big|\mathcal{F}_u\Big],
\end{aligned}$$

where we have taken the conditional expectation w.r.t. $\mathcal{F}_\tau$ inside the expectation. Since the process $\big(J(u,X_u^{t,x,\alpha},I_u^{\alpha};\alpha^0)-Y_u^{t,x,\alpha}\big)_{t\le u\le T}$ has right-continuous paths , by definition of τ we have $J(\tau,X_\tau^{t,x,\alpha},I_\tau^{\alpha},\alpha^0)-Y_\tau^{t,x,\alpha}\ge 0$ a.s., so that

$$\mathbf{1}_{\{u\le\tau\}}(Y_u^{t,x,\tilde{\alpha}}-Y_u^{t,x,\alpha}) \;\ge\; 0\,. \tag{2.8}$$

Noting that on $\{u\le\tau\}$ we have

$$\begin{aligned}
Y_u^{t,x,\alpha} &= Y_{u-}^{t,x,\alpha}+\Delta Y_u^{t,x,\alpha}\\
&\ge J(u,X_u^{t,x,\alpha},I_{u-}^{\alpha};\alpha_u^0)+c(X_u^{t,x,\alpha},I_{u-}^{\alpha},I_u^{\alpha})\\
&\ge -K(1+|X_u|)\,,
\end{aligned}$$

and since on $\{u > \tau\}$, $Y_u^{t,x,\tilde\alpha} = J(u, X_u^{t,x,\tilde\alpha}, I_u^{\tilde\alpha}; \alpha^0)$, from Lemma 2.1, it follows that $\tilde\alpha \in \tilde{\mathcal{A}}_{t,i}^{s,\tilde K}(x)$, for some $\tilde K$ not depending on (t,x). Furthermore taking $u = t$ in (2.8), we have $J(t,x,i;\tilde\alpha) \geq J(t,x,i;\alpha)$, and this proves the required assertion.

• *Step 4*. Finally we show that for each $\tilde K$, there exists some positive K s.t. $\tilde{\mathcal{A}}_{t,i}^{s,\tilde K}(x) \subset \mathcal{A}_{t,i}^{K}(x)$. We fix $\alpha \in \tilde{\mathcal{A}}_{t,i}^{s,\tilde K}(x)$. Applying Itô's formula to $|Y^{t,x,\alpha}|^2$ in (2.7), we have

$$\begin{aligned} |Y_t^{t,x,\alpha}|^2 + \int_t^T |Z_s^{t,x,\alpha}|^2 ds \;\; &= \;\; |g(X_T^{t,x,\alpha}, I_T^\alpha)|^2 + 2\int_t^T Y_s^{t,x,\alpha} f(X_s^{t,x,\alpha}, I_s^\alpha) ds \\ & \quad - 2\int_t^T Y_s^{t,x,\alpha} Z_s^{t,x,\alpha} dW_s - 2\int_t^T Y_s^{t,x,\alpha} dC_s^{t,x,\alpha} \; . \end{aligned}$$

Using **(Hl)** and the inequality $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$, we get

$$\begin{aligned} \int_t^T |Z_s^{t,x,\alpha}|^2 ds \;\; &\leq \;\; K\Big(1 + \sup_{s\in[t,T]} |X_s^{t,x,\alpha}|^2 + \sup_{s\in[t,T]} |Y_s^{t,x,\alpha}|^2 + |C_T^{t,x,\alpha} - C_t^{t,x,\alpha}| \sup_{s\in[t,T]} |Y_s^{t,x,\alpha}|\Big) \\ & \quad -2\int_t^T Y_s^{t,x,\alpha} Z_s^{t,x,\alpha} dW_s \; . \end{aligned} \tag{2.9}$$

Moreover, from (2.7), we have

$$\begin{aligned} |C_T^{t,x,\alpha} - C_t^{t,x,\alpha}|^2 \;\; &\leq \;\; K\Big(1 + \sup_{s\in[t,T]} |X_s^{t,x,\alpha}|^2 + \sup_{s\in[t,T]} |Y_s^{t,x,\alpha}|^2 \\ & \quad + \Big|\int_t^T Z_s^{t,x,\alpha} dW_s\Big|^2\Big) \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10) and using the inequality $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, for $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we obtain

$$\begin{aligned} \int_t^T |Z_s^{t,x,\alpha}|^2 ds \;\; &\leq \;\; K\Big((1+\varepsilon)\Big(1 + \sup_{s\in[t,T]} |X_s^{t,x,\alpha}|^2\Big) + \sup_{s\in[t,T]} |Y_s^{t,x,\alpha}|^2 \big(\varepsilon + \frac{1}{\varepsilon}\big) \\ & \quad + \varepsilon\Big|\int_t^T Z_s^{t,x,\alpha} dW_s\Big|^2\Big) - 2\int_t^T Y_s^{t,x,\alpha} Z_s^{t,x,\alpha} dW_s \; . \end{aligned}$$

Taking the expectation in the previous estimate, it follows from Lemma 2.1 and $\alpha \in \tilde{\mathcal{A}}_{t,i}^{s,\tilde K}(x)$ that

$$\begin{aligned} \mathbb{E}\Big[\int_t^T |Z_s^{t,x,\alpha}|^2 ds\Big] \;\; &\leq \;\; K\Big((1+\varepsilon)\Big(1 + \mathbb{E}\sup_{s\in[t,T]} |X_s^{t,x,\alpha}|^2\Big) + \big(\varepsilon + \frac{1}{\varepsilon}\big)\mathbb{E}\sup_{s\in[t,T]} |Y_s^{t,x,\alpha}|^2 \\ & \quad + \varepsilon\mathbb{E}\Big|\int_t^T Z_s^{t,x,\alpha} dW_s\Big| \\ &\leq \;\; K\Big((1+|x|^2)\big(1+\varepsilon+\frac{1}{\varepsilon}\big) + \varepsilon\mathbb{E}\Big[\Big(\int_t^T |Z_s^{t,x,\alpha}|^2 ds\Big)\Big]\Big) \; , \end{aligned}$$

Taking ε small enough, this yields

$$\mathbb{E}\Big[\int_t^T |Z_s^{t,x,\alpha}|^2 ds\Big] \;\; \leq \;\; K\big(1 + |x|^2\big) \; ,$$

Taking the expectation in (2.10), and using the previous inequality together with Lemma 2.1 and $\alpha \in \tilde{\mathcal{A}}_{t,i}^{s,\tilde{K}}(x)$, we get:

$$\mathbb{E}|C_T^{t,x,\alpha^*} - C_t^{t,x,\alpha^*}|^2 \quad \leq \quad K(1+|x|^2), \tag{2.11}$$

for some positive constant K not depending on (t,x,i). Since (τ_n) is strictly increasing, we know that at the initial time t, there is at most one decision time τ_1. Thus, from the linear growth condition on the switching cost, $\mathbb{E}[|C_t^{t,x,\alpha}|^2] \leq K(1+|x|^2)$, which implies with (2.11) that $\alpha \in \mathcal{A}_{t,i}^K(x)$, and this proves the required result. □

In the sequel of this paper, we shall assume that **(Hl)** and **(Hc)** stand in force.

3 Time discretization

We first consider a time discretization of $[0,T]$ with time step $h = T/m \leq 1$, and partition $\mathbb{T}_h = \{t_k = kh, k = 0,\ldots,m\}$. For $(t_k,i) \in \mathbb{T}_h \times \mathbb{I}_q$, we denote by $\mathcal{A}_{t_k,i}^h$ the set of admissible switching controls $\alpha = (\tau_n,\iota_n)_n$ in $\mathcal{A}_{t_k,i}$, such that τ_n are valued in $\{\ell h, \ell = k,\ldots,m\}$, and we consider the value functions for the discretized optimal switching problem:

$$\begin{aligned} v_i^h(t_k,x) \quad &= \quad \sup_{\alpha \in \mathcal{A}_{t_k,i}^h} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell})h + g(X_{t_m}^{t_k,x,\alpha}, I_{t_m}) \\ &\qquad\qquad - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha}, \iota_{n-1}, \iota_n)\Big], \end{aligned} \tag{3.1}$$

for $(t_k,i,x) \in \mathbb{T}_h \times \mathbb{I}_q \times \mathbb{R}^d$.

The next result provides an error analysis between the continuous-time optimal switching problem and its discrete-time version.

Theorem 3.1 *There exists a positive constant K (not depending on h) such that*

$$|v_i(t_k,x) - v_i^h(t_k,x)| \quad \leq \quad K(1+|x|^{5/2})\,(h\log(2T/h))^{1/2},$$

for all $(t_k,x,i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$.

If the cost functions c_{ij}, $i,j \in \mathbb{I}_q$, do not depend on x, then

$$|v_i(t_k,x) - v_i^h(t_k,x)| \quad \leq \quad K(1+|x|^{3/2})h^{1/2}$$

Remark 3.1 For optimal stopping problems, it is known that the approximation by the discrete-time version gives an error of order $h^{\frac{1}{2}}$, see e.g. [12] and [1]. We recover this rate of convergence for multiple switching problems when the switching costs do not depend on the state process. However, in the general case, the error is of order $(h\log(1/h))^{\frac{1}{2}}$. A rate of $h^{\frac{1}{2}-\varepsilon}$ was obtained in [5] in the case of uncontrolled state process X, and is improved and extended here when X may be influenced through its drift and diffusion coefficient by the switching control.

Before proving this Theorem, we need the three following lemmata. The first two deal with the regularity in time of the controlled diffusion uniformly in the control, and the third one deals with the regularity of the controlled diffusion with respect to the control.

Lemma 3.1 *There exists a constant K such that*

$$\sup_{\alpha\in\mathcal{A}_{t_k,i}} \max_{k\leq\ell\leq m-1} \Big\| \sup_{s\in[t_\ell,t_{\ell+1}]} \big|X_s^{t_k,x,\alpha} - X_{t_\ell}^{t_k,x,\alpha}\big| \Big\|_2 \quad\leq\quad K(1+|x|)h^{\frac{1}{2}} ,$$

for all $x\in\mathbb{R}^d$, $i\in\mathbb{I}_q$, $k=0,\ldots,n$.

Proof. From the definition of $X^{t,x,\alpha}$ in (2.3), we have for all $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{R}^d\times\mathbb{I}_q$ and $\alpha\in\mathcal{A}_{t_k,i}$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{u\in[t_\ell,s]}\big|X_u^{t,x,\alpha}-X_{t_\ell}^{t,x,\alpha}\big|^2\Big] \quad&\leq\quad K\Big(\mathbb{E}\Big[\Big(\int_{t_\ell}^s |b_{I_u}(X_u^{t,x,\alpha})|du\Big)^2\Big] \\ &\qquad + \mathbb{E}\Big[\sup_{u\in[t_\ell,s]}\Big|\int_{t_\ell}^u \sigma_{I_r}(X_r^{t,x,\alpha})dW_r\Big|^2\Big]\Big) ,\end{aligned}$$

for all $s\in[t_\ell,t_{\ell+1}]$. From BDG and Jensen inequalities, we then have

$$\mathbb{E}\Big[\sup_{u\in[t_\ell,s]}\big|X_u^{t,x,\alpha}-X_{t_\ell}^{t,x,\alpha}\big|^2\Big] \quad\leq\quad K\Big(\mathbb{E}\Big[\int_{t_\ell}^s \big|b_{I_u}(X_u^{t,x,\alpha})\big|^2du\Big] + \mathbb{E}\Big[\int_{t_\ell}^s \big|\sigma_{I_u}(X_u^{t,x,\alpha})\big|^2du\Big]\Big) ,$$

From the linear growth conditions on b_i and σ_i, for $i\in\mathbb{I}_q$, and Lemma 2.1, we conclude that

$$\mathbb{E}\Big[\sup_{s\in[t_\ell,t_{\ell+1}]}\big|X_s^{t,x,\alpha}-X_{t_\ell}^{t,x,\alpha}\big|^p\Big] \quad\leq\quad K_p(1+|x|^p)h.$$

□

Lemma 3.2 *There exists some positive constant K such that*

$$\sup_{\alpha\in\mathcal{A}_{t_k,i}} \Big\| \sup_{\substack{0\leq s,u\leq T\\ |s-u|\leq h}} \big|X_s^{t_k,x,\alpha}-X_u^{t_k,x,\alpha}\big| \Big\|_2 \quad\leq\quad K(1+|x|)\big(h\log(2T/h)\big)^{\frac{1}{2}} ,$$

Proof. This follows from Theorem 1 in [7], using the estimates from Lemma 2.1 and linear growth of b_i, σ_i. □

For a strategy $\alpha=(\tau_n,\iota_n)_n\in\mathcal{A}_{t_k,i}$ we denote by $\tilde\alpha=(\tilde\tau_n,\tilde\iota_n)_n$ the strategy of $\mathcal{A}^h_{t_k,i}$ defined by

$$\tilde\tau_n \;=\; \min\{t_\ell\in\mathbb{T}_h \;:\; t_\ell\geq\tau_n\} , \qquad \tilde\iota_n \;=\; \iota_n, \qquad n\in\mathbb{N}.$$

The strategy $\tilde\alpha$ can be seen as the approximation of the strategy α by an element of $\mathcal{A}^h_{t_k,i}$. We then have the following regularity result of the diffusion in the control α.

Lemma 3.3 *There exists some positive constant K such that*

$$\Big\|\sup_{s\in[t_k,T]}\big|X_s^{t_k,x,\alpha}-X_s^{t_k,x,\tilde\alpha}\big|\Big\|_2 \quad\leq\quad K\Big(\mathbb{E}[N(\alpha)^2]\Big)^{\frac{1}{4}}(1+|x|)h^{\frac{1}{2}} ,$$

for all $x\in\mathbb{R}^d$, $i\in\mathbb{I}_q$, $k=0,\ldots,n$ and $\alpha\in\mathcal{A}_{t_k,i}$.

Proof. From the definition of $X^{t,x,\alpha}$ and $X^{t,x,\tilde\alpha}$, for $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{R}_d\times\mathbb{I}_q$, $\alpha\in\mathcal{A}^K_{t_k,i}$, we have by BDG inequality:

$$\begin{aligned}\mathbb{E}\Big[\sup_{u\in[t_k,s]}\big|X^{t,x,\alpha}_u-X^{t,x,\tilde\alpha}_u\big|^2\Big] &\le K\Big(\mathbb{E}\Big[\int_{t_k}^s\big|b(X^{t,x,\alpha}_u,I_u)-b(X^{t,x,\tilde\alpha}_u,\tilde I_u)\big|^2du\Big]\\ &\quad+\mathbb{E}\Big[\int_{t_k}^s\big|\sigma(X^{t,x,\alpha}_u,I_u)-\sigma(X^{t,x,\tilde\alpha}_u,\tilde I_u)\big|^2du\Big]\Big)\,,\end{aligned}$$

for all $s\in[t_k,T]$. Then using Lipschitz property of b_i and σ_i for $i\in\mathbb{I}_q$ we get:

$$\begin{aligned}\mathbb{E}\Big[\sup_{u\in[t_k,s]}\big|X^{t,x,\alpha}_s-X^{t,x,\tilde\alpha}_s\big|^2\Big] &\le K\Big(\mathbb{E}\Big[\int_{t_k}^s\big|X^{t,x,\alpha}_u-X^{t,x,\tilde\alpha}_u\big|^2du\Big]\\ &\quad+\mathbb{E}\Big[\int_{t_k}^s\big|b(X^{t,x,\alpha}_u,I_u)-b(X^{t,x,\alpha}_u,\tilde I_u)\big|^2du\Big]\\ &\quad+\mathbb{E}\Big[\int_{t_k}^s\big|\sigma(X^{t,x,\alpha}_u,I_u)-\sigma(X^{t,x,\alpha}_u,\tilde I_u)\big|^2du\Big]\Big)\\ &\le K\Big(\mathbb{E}\Big[\int_{t_k}^s\sup_{r\in[t_k,u]}\big|X^{t,x,\alpha}_r-X^{t,x,\tilde\alpha}_r\big|^2du\Big] \qquad (3.2)\\ &\quad+\mathbb{E}\Big[\big(\sup_{u\in[t_k,T]}\big|X^{t,x,\alpha}_u\big|^2+1\big)\int_{t_k}^s\mathbf{1}_{I_s\neq\tilde I_s}ds\Big]\Big)\,,\end{aligned}$$

for all $s\in[t_k,T]$. From the definition of $\tilde\alpha$ we have

$$\int_{t_k}^s\mathbf{1}_{I_s\neq\tilde I_s}ds \;\le\; N(\alpha)h\,,$$

which gives with (3.2), Lemma 2.1, Remark 2.1 and Hölder inequality:

$$\begin{aligned}\mathbb{E}\Big[\sup_{u\in[t_k,s]}\big|X^{t,x,\alpha}_u-X^{t,x,\tilde\alpha}_u\big|^2\Big] &\le K\Big(\mathbb{E}\Big[\int_{t_k}^s\sup_{r\in[t_k,u]}\big|X^{t,x,\alpha}_r-X^{t,x,\tilde\alpha}_r\big|^2du\Big]\\ &\quad+\big(\mathbb{E}[N(\alpha)^2]\big)^{\frac12}(1+|x|^2)h\Big),\end{aligned}$$

for all $s\in[t_k,T]$. We conclude with Gronwall's Lemma. □

We are now ready to prove the convergence result for the time discretization of the optimal switching problem.

Proof of Theorem 3.1. We introduce the auxiliary function $\tilde v^h_i$ defined by

$$\tilde v^h_i(t_k,x) \;=\; \sup_{\alpha\in\mathcal{A}^h_{t_k,i}}\mathbb{E}\Big[\int_{t_k}^T f(X^{t_k,x,\alpha}_s,I_s)ds+g(X^{t_k,x,\alpha}_T,I_T)-\sum_{n=1}^{N(\alpha)}c(X^{t_k,x,\alpha}_{\tau_n},\iota_{n-1},\iota_n)\Big]\,,$$

for all $(t_k,x)\in\mathbb{T}_h\times\mathbb{R}^d$. We then write

$$|v_i(t_k,x)-v^h_i(t_k,x)| \;\le\; |v_i(t_k,x)-\tilde v^h_i(t_k,x)|+|\tilde v^h_i(t_k,x)-v^h_i(t_k,x)|\,,$$

and study each of the two terms in the right-hand side.

• Let us investigate the first term. By definition of the approximating strategy $\tilde\alpha = (\tilde\tau_n, \tilde\iota_n)_n \in \mathcal{A}^h_{t_k,i}$ of $\alpha \in \mathcal{A}_{t_k,i}$, we see that the auxiliary value function $\tilde v_i^h$ may be written as

$$\tilde v_i^h(t_k,x) \;=\; \sup_{\alpha\in\mathcal{A}_{t_k,i}} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\tilde\alpha},\tilde I_s)ds + g(X_T^{t_k,x,\tilde\alpha},\tilde I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tilde\tau_n}^{t_k,x,\tilde\alpha},\tilde\iota_{n-1},\tilde\iota_n)\Big],$$

where $\tilde I$ is the indicator of the regime value associated to $\tilde\alpha$. Fix now a positive number $\bar K$ s.t. relation (2.6) in Proposition 2.1 holds, and observe that

$$\begin{aligned}
& \sup_{\alpha\in\mathcal{A}^{\bar K}_{t_k,i}(x)} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\tilde\alpha},\tilde I_s)ds + g(X_T^{t_k,x,\tilde\alpha},\tilde I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tilde\tau_n}^{t_k,x,\tilde\alpha},\tilde\iota_{n-1},\tilde\iota_n)\Big] \\
\leq\;& \tilde v_i^h(t_k,x) \;\leq\; v_i(t_k,x) \\
=\;& \sup_{\alpha\in\mathcal{A}^{\bar K}_{t_k,i}(x)} \mathbb{E}\Big[\int_{t_k}^T f(X_s^{t_k,x,\alpha},I_s)ds + g(X_T^{t_k,x,\alpha},I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n)\Big].
\end{aligned}$$

We then have

$$|v_i(t_k,x) - \tilde v_i^h(t_k,x)| \;\leq\; \sup_{\alpha\in\mathcal{A}^{\bar K}_{t_k,i}(x)} \Big[\Delta^1_{t_k,x}(\alpha) + \Delta^2_{t_k,x}(\alpha)\Big], \tag{3.3}$$

with

$$\begin{aligned}
\Delta^1_{t_k,x}(\alpha) &= \mathbb{E}\Big[\int_{t_k}^T \big|f(X_s^{t_k,x,\alpha},I_s) - f(X_s^{t_k,x,\tilde\alpha},\tilde I_s)\big|ds + \big|g(X_T^{t_k,x,\alpha},I_T) - g(X_T^{t,x,\tilde\alpha},\tilde I_T)\big|\Big], \\
\Delta^2_{t_k,x}(\alpha) &= \mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|c(X_{\tau_n}^{t_k,x,\alpha},\iota_{n-1},\iota_n) - c(X_{\tilde\tau_n}^{t_k,x,\tilde\alpha},\tilde\iota_{n-1},\tilde\iota_n)\big|\Big].
\end{aligned}$$

Under **(Hl)**, and by definition of $\tilde\alpha$, there exists some positive constant K s.t.

$$\begin{aligned}
\Delta^1_{t_k,x}(\alpha) &\leq K\Big(\sup_{s\in[t_k,T]}\mathbb{E}\Big[\big|X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde\alpha}\big|\Big] + \mathbb{E}\Big[\big(\sup_{s\in[t_k,T]}\big|X_s^{t_k,x,\alpha}\big| + 1\big)\int_{t_k}^T \mathbf{1}_{I_s\neq\tilde I_s}ds\Big]\Big). \\
&\leq K\Big(\sup_{s\in[t_k,T]}\mathbb{E}\Big[\big|X_s^{t_k,x,\alpha} - X_s^{t_k,x,\tilde\alpha}\big|\Big] \\
&\qquad + \Big(1 + \Big\|\sup_{s\in[t_k,T]}\big|X_s^{t_k,x,\alpha}\big|\Big\|_2\Big)\Big(\mathbb{E}\Big[\int_{t_k}^T \mathbf{1}_{I_s\neq\tilde I_s}ds\Big]\Big)^{\frac12}\Big),
\end{aligned} \tag{3.4}$$

by Cauchy-Schwarz inequality. For $\alpha \in \mathcal{A}^K_{t_k,i}(x)$, we have by Remark 2.1

$$\mathbb{E}\Big[\int_{t_k}^T \mathbf{1}_{I_s\neq\tilde I_s}ds\Big] \;\leq\; h\mathbb{E}\Big[N(\alpha)\Big] \;\leq\; \eta\bar K_1(1+|x|)h,$$

for some positive constant $\eta > 0$. By using this last estimate together with Lemmata 2.1 and 3.3 into (3.4), we obtain the existence of some constant K s.t.

$$\sup_{\alpha\in\mathcal{A}^{\bar K}_{t_k,i}(x)} \Delta^1_{t_k,x}(\alpha) \;\leq\; K(1+|x|^{3/2})h^{\frac12}, \tag{3.5}$$

for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$.

We now turn to the term $\Delta^2_{t,x}(\alpha)$. Under **(Hl)**, and by definition of $\tilde{\alpha}$, there exists some positive constant K s.t.

$$\begin{aligned}
\Delta^2_{t_k,x}(\alpha) &\leq K\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|X^{t_k,x,\alpha}_{\tau_n} - X^{t_k,x,\tilde{\alpha}}_{\tilde{\tau}_n}\big|\Big] \\
&\leq K\Big(\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|X^{t_k,x,\alpha}_{\tau_n} - X^{t_k,x,\alpha}_{\tilde{\tau}_n}\big|\Big] + \mathbb{E}\Big[N(\alpha)\sup_{s\in[t_k,T]}\big|X^{t_k,x,\alpha}_s - X^{t_k,x,\tilde{\alpha}}_s\big|\Big]\Big) \\
&\leq K\Big(\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|X^{t_k,x,\alpha}_{\tau_n} - X^{t_k,x,\alpha}_{\tilde{\tau}_n}\big|\Big] \\
&\qquad + \Big\|N(\alpha)\Big\|_2\Big\|\sup_{s\in[t_k,T]}\big|X^{t_k,x,\alpha}_s - X^{t_k,x,\tilde{\alpha}}_s\big|\Big\|_2\Big),
\end{aligned} \tag{3.6}$$

by Cauchy-Schwarz inequality. For $\alpha \in \mathcal{A}^K_{t_k,i}(x)$ with Remark 2.1, and from Lemma 3.3, we get the existence of some positive constant K s.t.

$$\Big\|N(\alpha)\Big\|_2\Big\|\sup_{s\in[t_k,T]}\big|X^{t_k,x,\alpha}_s - X^{t_k,x,\tilde{\alpha}}_s\big|\Big\|_2 \leq K(1+|x|^{5/2})h^{\frac{1}{2}}\,. \tag{3.7}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|X^{t_k,x,\alpha}_{\tau_n} - X^{t_k,x,\alpha}_{\tilde{\tau}_n}\big|\Big] &\leq \mathbb{E}\Big[N(\alpha)\sup_{\substack{0\leq s,u\leq T\\|s-u|\leq h}}\big|X^{t_k,x,\alpha}_s - X^{t_k,x,\alpha}_u\big|\Big] \\
&\leq \Big\|N(\alpha)\Big\|_2\Big\|\sup_{\substack{0\leq s,u\leq T\\|s-u|\leq h}}\big|X^{t_k,x,\alpha}_s - X^{t_k,x,\alpha}_u\big|\Big\|_2
\end{aligned}$$

by Cauchy-Schwarz inequality. For $\alpha \in \mathcal{A}^{\bar{K}}_{t_k,i}(x)$, by Lemma 3.2, this yields the existence of some positive constant K s.t.

$$\mathbb{E}\Big[\sum_{n=1}^{N(\alpha)} \big|X^{t_k,x,\alpha}_{\tau_n} - X^{t_k,x,\alpha}_{\tilde{\tau}_n}\big|\Big] \leq K(1+|x|^2)\,(h\log(2T/h))^{1/2}\,. \tag{3.8}$$

By plugging (3.7) and (3.8) into (3.6), we then get

$$\Delta^2_{t,x}(\alpha) \leq K(1+|x|^2)\,(h\log(2T/h))^{1/2}\,. \tag{3.9}$$

Combining (3.5) and (3.9), we obtain with (3.3)

$$|v_i(t_k,x) - \tilde{v}^h_i(t_k,x)| \leq K(1+|x|^2)\,(h\log(2T/h))^{1/2}\,.$$

In the case where c does not depend on the variable x, we have $\Delta^2_{t,x}(\alpha) = 0$, and so by (3.3), (3.5):

$$|v_i(t_k,x) - \tilde{v}^h_i(t_k,x)| \leq K(1+|x|^{3/2})h^{\frac{1}{2}}\,.$$

• For the second term, we have by definition of v_i^h and $\tilde{v}_i^h$:

$$|\tilde{v}_i^h(t_k,x) - v_i^h(t_k,x)| \;\leq\; \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1}\int_{t_\ell}^{t_{\ell+1}} \big|f(X_s^{t,x,\alpha},I_s) - f(X_{t_\ell}^{t,x,\alpha},I_s)\big|ds\Big],$$

since $I_s = I_{t_\ell}$ on $[t_\ell,t_{\ell+1})$. Under **(Hl)**, we get

$$|\tilde{v}_i^h(t_k,x) - v_i^h(t_k,x)| \;\leq\; K \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \max_{k\leq\ell\leq m-1} \sup_{s\in[t_\ell,t_{\ell+1}]} \mathbb{E}\Big[\big|X_s^{t,x,\alpha} - X_{t_\ell}^{t,x,\alpha}\big|\Big],$$

for some positive constant K, and by Lemma 3.1, this shows that

$$|\tilde{v}_i^h(t_k,x) - v_i^h(t_k,x)| \;\leq\; K(1+|x|)h^{\frac{1}{2}}\ .$$

□

In a second step, we approximate the continuous-time (controlled) diffusion by a discrete-time (controlled) Markov chain following an Euler type scheme. For any $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{R}^d\times\mathbb{I}_q$, $\alpha\in\mathcal{A}^h_{t_k,i}$, we introduce $(\bar{X}_{t_\ell}^{h,t_k,x,\alpha})_{k\leq\ell\leq m}$ defined by:

$$\bar{X}_{t_k}^{h,t_k,x,\alpha} \;=\; x, \qquad \bar{X}_{t_{\ell+1}}^{h,t_k,x,\alpha} \;=\; F^h_{I_{t_\ell}}(\bar{X}_{t_\ell}^{h,t_k,x,\alpha},\vartheta_{\ell+1}), \quad k\leq\ell\leq m-1,$$

where

$$F_i^h(x,\vartheta_{k+1}) \;=\; x + b_i(x)h + \sigma_i(x)\sqrt{h}\,\vartheta_{k+1},$$

and $\vartheta_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$, $k=0,\ldots,m-1$, are iid, $\mathcal{N}(0,I_d)$-distributed, independent of $\mathcal{F}_{t_k}$. Similarly as in Lemma 2.1, we have the L^p-estimate:

$$\sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \Big\|\max_{\ell=k,\ldots,m}\big|\bar{X}_{t_\ell}^{h,t_k,x,\alpha}\big|\Big\|_p \;\leq\; K_p(1+|x|), \tag{3.10}$$

for some positive constant K_p, not depending on (h,t_k,x,i). Moreover, one can also derive the standard estimate for the Euler scheme, as e.g. in section 10.2 of [11]:

$$\sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \Big\|\max_{\ell=k,\ldots,m}\big|X_{t_\ell}^{t_k,x,\alpha} - \bar{X}_{t_\ell}^{h,t_k,x,\alpha}\big|\Big\|_p \;\leq\; K_p(1+|x|)\sqrt{h}. \tag{3.11}$$

We then associate to the Euler controlled Markov chain, the value functions $\bar{v}_i^h$, $i\in\mathbb{I}_q$, for the optimal switching problem:

$$\begin{aligned}\bar{v}_i^h(t_k,x) \;=\; \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[&\sum_{\ell=k}^{m-1} f(\bar{X}_{t_\ell}^{h,t_k,x,\alpha},I_{t_\ell})h + g(\bar{X}_{t_m}^{h,t_k,x,\alpha},I_{t_m}) \\ &- \sum_{n=1}^{N(\alpha)} c(\bar{X}_{\tau_n}^{h,t_k,x,\alpha},\iota_{n-1},\iota_n)\Big].\end{aligned} \tag{3.12}$$

The next result provides the error analysis between v_i^h by $\bar{v}_i^h$, and thus of the continuous time optimal switching problem v_i by its Euler discrete-time approximation $\bar{v}_i^h$.

Theorem 3.2 *There exists a constant K (not depending on h) such that*

$$\left|v_i^h(t_k,x) - \bar v_i^h(t_k,x)\right| \;\; \leq \;\; K(1+|x|^2)\sqrt{h}, \tag{3.13}$$

for all $(t_k,x,i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$.

Remark 3.2 The above theorem combined with Theorem 3.1 gives the rate of convergence for the approximation of the continuous time optimal switching problem by its Euler discrete-time version: there exists a positive constant K s.t.

$$|v_i(t_k,x) - \bar v_i^h(t_k,x)| \;\; \leq \;\; K(1+|x|^{5/2})\big(h\log(2T/h)\big)^{\frac{1}{2}}, \tag{3.14}$$

for all $(t_k,x,i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$. Moreover if the cost functions c_{ij}, $i,i \in \mathbb{I}_q$, do not depend on x, then

$$|v_i(t_k,x) - \bar v_i^h(t_k,x)| \;\; \leq \;\; K(1+|x|^2)h^{\frac{1}{2}},$$

Proof of Theorem 3.2.

• *Step 1.* For $(t_k,x,i) \in \mathbb{T}_h \times \mathbb{R}^d \times \mathbb{I}_q$, and $\alpha \in \mathcal{A}^h_{t_k,i}$ we denote by

$$J^h(t_k,x,i;\alpha) \;\; = \;\; \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell})h + g(X_{t_m}^{t_k,x,\alpha}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}^{t_k,x,\alpha}, \iota_{n-1},\iota_n)\Big],$$

so that $v_i^h(t_{,k}\,,x) = \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} J^h(t_k,x,i,\alpha)$. Given $\alpha \in \mathcal{A}^h_{t_k,i}$, let us define $F_\ell^\alpha = f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell}^\alpha)$, $c_\ell^\alpha = c(X_{t_\ell}^{t_k,x,\alpha}, I_{t_{\ell-1}}^\alpha, I_{t_\ell}^\alpha)$ and $Y_\ell^\alpha = \mathbb{E}\big[\sum_{j=\ell}^m \big(hF_j^\alpha - c_j^\alpha\big)|\mathcal{F}_{t_\ell}\big]$, for $\ell = k,\ldots,m$. Consider the stopping time

$$\tau \;\; = \;\; \inf\{t_\ell \geq t_k \;:\; J^h(t_\ell, X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell}^\alpha;\alpha^0) \geq Y_\ell^\alpha\}\;,$$

where α^0 is the strategy with no switches, and define $\tilde\alpha = (\tilde\tau_n,\iota_n)$, with

$$\tilde\tau_n \;\; = \;\; \tau_n \mathbf{1}_{\{\tau_n\leq\tau\}} + \infty\mathbf{1}_{\{\tau_n>\tau\}}.$$

As in the proof of Proposition 2.1, we easily check that

$$Y_k^{\tilde\alpha} \;\; \geq \;\; Y_k^\alpha, \tag{3.15}$$

and

$$Y_\ell^{\tilde\alpha} \;\; \geq \;\; J(t_\ell, X_{t_\ell}^{t_k,x,\tilde\alpha}, I_{t_\ell}^{\tilde\alpha};\alpha^0)\;, \tag{3.16}$$

for all $\ell = k,\ldots,m$. From (3.16) and the estimates on $X_{t_\ell}^{t_k,x,\alpha}$ in Lemma 2.1, we know that

$$\mathbb{E}\Big[\sup_{k\leq\ell\leq m}\big(|Y_\ell^{\tilde\alpha}|^2 + |F_\ell^{\tilde\alpha}|^2 + |c_\ell^{\tilde\alpha}|^2\big)\Big] \;\; \leq \;\; K(1+|x|^2)\;, \tag{3.17}$$

for some positive constant K. Moreover, by definition, we have:

$$Y_\ell^{\tilde\alpha} \;\; = \;\; \mathbb{E}\left[Y_{\ell+1}^{\tilde\alpha}|\mathcal{F}_{t_\ell}\right] + hF_\ell - c_\ell, \quad \ell = k,\ldots,m-1.$$

Letting $\Delta M^{\tilde{\alpha}}_{\ell+1} := Y^{\tilde{\alpha}}_{\ell+1} - \mathbb{E}[Y^{\tilde{\alpha}}_{\ell+1}|\mathcal{F}_{t_\ell}]$, we obtain in particular

$$\sum_{\ell=k}^{m-1} c^{\tilde{\alpha}}_\ell \;=\; h\sum_{\ell=k}^{m-1} F^{\tilde{\alpha}}_\ell - \sum_{\ell=k}^{m-1} \Delta M^{\tilde{\alpha}}_{\ell+1} + (Y^{\tilde{\alpha}}_m - Y^{\tilde{\alpha}}_k)\,,$$

and so by (3.17)

$$\begin{aligned}\mathbb{E}\Big|\sum_{\ell=k}^{m} c^{\tilde{\alpha}}_\ell\Big|^2 \;&\leq\; K(1+|x|^2) + 3\,\mathbb{E}\left[\left(\sum_{\ell=k}^{m-1}\Delta M^{\tilde{\alpha}}_{\ell+1}\right)^2\right]\\ &=\; K(1+|x|^2) + 3\,\mathbb{E}\left[\sum_{\ell=k}^{m-1}|\Delta M^{\tilde{\alpha}}_{\ell+1}|^2\right]. \end{aligned} \tag{3.18}$$

Now by writing that

$$\begin{aligned}|Y^{\tilde{\alpha}}_m|^2 - |Y^{\tilde{\alpha}}_k|^2 \;&=\; \sum_{\ell=k}^{m-1}\left(|Y^{\tilde{\alpha}}_{\ell+1}|^2 - |Y^{\tilde{\alpha}}_\ell|^2\right) \;=\; \sum_{\ell=k}^{m-1}(Y^{\tilde{\alpha}}_{\ell+1} - Y^{\tilde{\alpha}}_\ell)(Y^{\tilde{\alpha}}_{\ell+1} + Y^{\tilde{\alpha}}_\ell)\\ &=\; \sum_{\ell=k}^{m-1}(\Delta M^{\tilde{\alpha}}_{\ell+1} - hF^{\tilde{\alpha}}_\ell + c^{\tilde{\alpha}}_\ell)(2Y^{\tilde{\alpha}}_\ell + \Delta M^{\tilde{\alpha}}_{\ell+1} - hF^{\tilde{\alpha}}_\ell + c^{\tilde{\alpha}}_\ell),\end{aligned}$$

we get

$$\begin{aligned}\sum_{\ell=k}^{m-1}|\Delta M^{\tilde{\alpha}}_{\ell+1}|^2 \;&=\; |Y^{\tilde{\alpha}}_m|^2 - |Y^{\tilde{\alpha}}_0|^2 - \sum_{\ell=0}^{m-1} hF^{\tilde{\alpha}}_\ell(hF^{\tilde{\alpha}}_\ell - 2Y^{\tilde{\alpha}}_\ell - 2c^{\tilde{\alpha}}_\ell) - 2\sum_{\ell=0}^{m-1} c^{\tilde{\alpha}}_\ell Y^{\tilde{\alpha}}_\ell\\ &\quad - \sum_{\ell=0}^{m-1}\Delta M^{\tilde{\alpha}}_{\ell+1}(2Y^{\tilde{\alpha}}_\ell - 2hF^{\tilde{\alpha}}_\ell + 2c^{\tilde{\alpha}}_\ell) - \sum_{\ell=0}^{m-1}|c^{\tilde{\alpha}}_\ell|^2.\end{aligned}$$

Since $\mathbb{E}\Big[\Delta M^{\tilde{\alpha}}_{\ell+1}|\mathcal{F}_{t_\ell}\Big] = 0$, this shows that

$$\begin{aligned}\mathbb{E}\Big[\sum_{\ell=k}^{m-1}|\Delta M^{\tilde{\alpha}}_{\ell+1}|^2\Big] \;&\leq\; \mathbb{E}\Big[|Y^{\tilde{\alpha}}_m|^2 - \sum_{\ell=0}^{m-1} hF^{\tilde{\alpha}}_\ell(hF^{\tilde{\alpha}}_\ell - 2Y^{\tilde{\alpha}}_\ell - 2c^{\tilde{\alpha}}_\ell) - 2\sum_{\ell=0}^{m-1} c^{\tilde{\alpha}}_\ell Y^{\tilde{\alpha}}_\ell\Big]\\ &\leq\; K(1+|x|^2) + 2\mathbb{E}\Big[\Big|\sum_{\ell=0}^{m-1} c^{\tilde{\alpha}}_\ell Y^{\tilde{\alpha}}_\ell\Big|\Big],\end{aligned} \tag{3.19}$$

where we used again (3.17). Now since $c_\ell \geq 0$,

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{\ell=0}^{m-1} c^{\tilde{\alpha}}_\ell Y^{\tilde{\alpha}}_\ell\Big|\Big] \;&\leq\; \mathbb{E}\Big[\Big(\sum_{\ell=0}^{m-1} c_\ell\Big)\sup_{k\leq\ell\leq m-1}|Y^{\tilde{\alpha}}_\ell|\Big]\\ &\leq\; \varepsilon\mathbb{E}\Big[\sum_{\ell=k}^{m-1}|\Delta M^{\tilde{\alpha}}_{\ell+1}|^2\Big] + K\big(1+\frac{1}{\varepsilon}\big)(1+|x|^2),\end{aligned}$$

for all $\varepsilon > 0$, by (3.17), (3.18) and Cauchy-Schwarz inequality. Hence taking ε small enough and plugging this estimate into (3.19), we obtain

$$\mathbb{E}\Big[\sum_{\ell=k}^{m-1}|\Delta M^{\tilde{\alpha}}_{\ell+1}|^2\Big] \;\leq\; K(1+|x|^2).$$

Using (3.18) one more time and recalling that $N(\tilde\alpha) \leq \eta \sum_\ell c_\ell^{\tilde\alpha}$ for some $\eta > 0$ under the uniformly lower bound condition in **(Hc)**, we thus obtain

$$\mathbb{E}\big|N(\tilde\alpha)\big|^2 \quad \leq \quad K(1+|x|^2). \tag{3.20}$$

Combining this last inequality with (3.15), we get that the supremum in the definition (3.1) of $v_i^h(t_k,x)$ can be taken over $\mathcal{A}_{t_k,i}^{h,K}(x) = \big\{\alpha \in \mathcal{A}_{t_k,i}^h \text{ s.t. } \mathbb{E}|N(\alpha)|^2 \leq K(1+|x|^2)\big\}$. Using the same argument with $\bar X^{t_k,x,\alpha}$ instead of $X^{t_k,x,\alpha}$ and estimate (3.10) on $\big\|\bar X_{t_\ell}^{h,t_k,x,\alpha}\big\|_2$ we also get that the supremum in the definition (3.12) $\bar v_i^h(t_k,x)$ can be taken over $\mathcal{A}_{t_k,i}^{h,K}(x)$.

• *Step 2.* Now, for any $\alpha \in \mathcal{A}_{t_k,i}^{h,K}(x)$, we have under **(Hl)** and by Cauchy-Schwarz inequality

$$\begin{aligned}
& \mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(X_{t_\ell}^{t_k,x,\alpha}, I_{t_\ell}) - f(\bar X_{t_\ell}^{h,t_k,x,\alpha}, I_{t_\ell})\big| + \big|g(X_{t_m}^{t_k,x,\alpha}, I_{t_m}) - g(\bar X_{t_m}^{h,t_k,x,\alpha}, I_{t_m})\big| \\
& \quad + \sum_{n=1}^{N(\alpha)} \big|c(X_{\tau_n}^{t_k,x,\alpha}, \iota_{n-1}, \iota_n) - c(\bar X_{\tau_n}^{h,t_k,x,\alpha}, \iota_{n-1}, \iota_n)\big|\Big] \\
\leq \quad & K\mathbb{E}\Big[(1+N(\alpha))\big(\sup_{k\leq\ell\leq m}\big|X_{t_\ell}^{t_k,x,\alpha} - \bar X_{t_\ell}^{h,t_k,x,\alpha}\big|\big)\Big] \\
\leq \quad & K(1+|x|)\Big\|\sup_{k\leq\ell\leq m}\big|X_{t_\ell}^{t_k,x,\alpha} - \bar X_{t_\ell}^{h,t_k,x,\alpha}\big|\Big\|_2 \\
\leq \quad & K(1+|x|^2)\sqrt{h},
\end{aligned} \tag{3.21}$$

by (3.11). Taking the supremum over $\alpha \in \mathcal{A}_{t_k,i}^{h,K}(x)$ into (3.21), this shows that

$$\big|v_i^h(t_k,x) - \bar v_i^h(t_k,x)\big| \quad \leq \quad K(1+|x|^2)\sqrt{h}.$$

□

4 Approximation schemes by optimal quantization

In this section, for a fixed time discretization step h, we focus on a computational approximation for the value functions $\bar v_i^h$, $i \in \mathbb{I}_q$, defined in (3.12). To alleviate notations, we shall often omit the dependence on h in the superscripts, and write e.g. $\bar v_i = \bar v_i^h$. The corresponding dynamic programming relation for $\bar v_i$ is written in the backward induction:

$$\begin{aligned}
\bar v_i(t_m,x) &= g_i(x), \\
\bar v_i(t_k,x) &= \max\Big\{\mathbb{E}\big[\bar v_i(t_{k+1}, \bar X_{t_{k+1}}^{t_k,x,i})\big] + f_i(x)h\ ,\ \max_{j\neq i}[\bar v_j(t_k,x) - c_{ij}(x)]\Big\},
\end{aligned}$$

for $k = 0,\ldots,m-1$, $(i,x) \in \mathbb{I}_q \times \mathbb{R}^d$, where $\bar X^{t_k,x,i}$ is the solution to the Euler scheme:

$$\bar X_{t_{k+1}}^{t_k,x,i} \quad = \quad F_i^h(x,\vartheta_{k+1}) \ :=\ x + b_i(x)h + \sigma_i(x)\sqrt{h}\,\vartheta_{k+1}.$$

Observe that under the triangular condition on the switching costs c_{ij} in **(Hc)**, these backward relations can be written as an explicit discrete-time scheme. Indeed, if $\bar v_i(t_k,x) = \bar v_j(t_k,x) - c_{ij}(x)$ for some $j \neq i$, we then have for $l \neq i,j$,

$$\begin{aligned}
\bar v_j(t_k,x) - c_{ij}(x) \ = \ \bar v_i(t_k,x) \quad &\geq \quad \bar v_l(t_k,x) - c_{il}(x) \\
&> \quad \bar v_l(t_k,x) - c_{ij}(x) - c_{jl}(x),
\end{aligned}$$

so that $\bar{v}_j(t_k,x) > \bar{v}_l(t_k,x) - c_{jl}(x)$. By positivity of the switching costs, we also have

$$\bar{v}_j(t_k,x) = \bar{v}_i(t_k,x) + c_{ij}(x) > \bar{v}_i(t_k,x) - c_{ji}(x).$$

It follows that

$$\bar{v}_j(t_k,x) \quad = \quad \mathbb{E}\big[\bar{v}_j(t_{k+1},\bar{X}^{t_k,x,j}_{t_{k+1}})\big] + f_j(x)h,$$

and (recalling that $c_{ii}(\cdot) = 0$), the backward induction may be rewritten as

$$\begin{aligned} \bar{v}_i(t_m,x) &= g_i(x) && (4.1) \\ \bar{v}_i(t_k,x) &= \max_{j\in\mathbb{I}_q}\Big\{\mathbb{E}\big[\bar{v}_j(t_{k+1},\bar{X}^{t_k,x,j}_{t_{k+1}})\big] + f_j(x)h - c_{ij}(x)\Big\}, && (4.2) \end{aligned}$$

for $k = 0,\dots,m-1$, $(i,x) \in \mathbb{I}_q \times \mathbb{R}^d$. Next, the practical implementation for this scheme requires a computational approximation of the expectations arising in the above dynamic programming formulae, and a space discretization for the state process X valued in $\mathbb{R}^d$. We shall propose two numerical approximations schemes by optimal quantization methods, the second one in the particular case where the state process X is not controlled by the switching control.

4.1 A Markovian quantization method

Let $\mathbb{X}$ be a bounded lattice grid on $\mathbb{R}^d$ with step δ/d and size R, namely $\mathbb{X} = (\delta/d)\mathbb{Z}^d \cap B(0,R) = \{x \in \mathbb{R}^d : x = (\delta/d)z$ for some $z \in \mathbb{Z}^d$, and $|x| \le \mathbb{R}\}$. We then denote by $\mathrm{Proj}_{\mathbb{X}}$ the projection on the grid $\mathbb{X}$ according to the closest neighbour rule, which satisfies

$$|x - \mathrm{Proj}_{\mathbb{X}}(x)| \quad \le \quad \max(|x| - R, 0) + \delta, \quad \forall x \in \mathbb{R}^d. \qquad (4.3)$$

At each time step $t_k \in \mathbb{T}_h$, and point space-grid $x \in \mathbb{X}$, we have to compute in (4.2) expectations in the form $\mathbb{E}\Big[\varphi(\bar{X}^{t_k,x,i}_{t_{k+1}})\Big]$, for $\varphi(.) = \bar{v}^h_i(t_{k+1},.)$, $i \in \mathbb{I}_q$. We shall then use an optimal quantization for the Gaussian random variable ϑ_{k+1}, which consists in approximating the distribution of $\vartheta \rightsquigarrow \mathcal{N}(0,I_d)$ by the discrete law of a random variable $\hat{\vartheta}$ of support N points w_l, $l = 1,\dots,N$, in $\mathbb{R}^d$, and defined as the projection of ϑ on the grid $\{w_1,\dots,w_N\}$ following the closest neighbor rule. The grid $\{w_1,\dots,w_N\}$ is optimized in order to minimize the distorsion error, i.e. the quadratic L^2-norm $\big\|\vartheta - \hat{\vartheta}\big\|_2$. This optimal grid and the associated weights $\{\pi_1,\dots,\pi_N\}$ are downloaded from the website: "http://www.quantize.maths-fi.com/downloads". We refer to the survey article [15] for more details on the theoretical and computational aspects of optimal quantization methods. In the vein of [16], we introduce the quantized Euler scheme:

$$\hat{X}^{t_k,x,i}_{t_{k+1}} \quad = \quad \mathrm{Proj}_{\mathbb{X}}(F^h_i(x,\hat{\vartheta})),$$

and define the value functions $\hat{v}_i$ on $\mathbb{T}_m \times \mathbb{X}$, $i \in \mathbb{I}_q$ in backward induction by

$$\begin{aligned} \hat{v}_i(t_m,x) &= g_i(x) \\ \hat{v}_i(t_k,x) &= \max_{j\in\mathbb{I}_q}\Big\{\mathbb{E}\big[\hat{v}_j(t_{k+1},\hat{X}^{t_k,x,j}_{t_{k+1}})\big] + f_j(x)h - c_{ij}(x)\Big\},\ k = 0,\dots,m-1. \end{aligned}$$

This numerical scheme can be computed explicitly according to the following recursive algorithm:

$$\begin{aligned}
\hat{v}_i(t_m, x) &= g_i(x), \quad (x,i) \in \mathbb{X} \times \mathbb{I}_q \\
\hat{v}_i(t_k, x) &= \max_{j \in \mathbb{I}_q} \Big[\sum_{l=1}^{N} \pi_l \, \hat{v}_j\big(t_{k+1}, \mathrm{Proj}_{\mathbb{X}}(F_j^h(x, w_l))\big) + f_j(x)h - c_{ij}(x) \Big], \quad (x,i) \in \mathbb{X} \times \mathbb{I}_q,
\end{aligned}$$

for $k = 0, \ldots, m-1$. At each time step, we need to make $O(N)$ computations for each point of the grid $\mathbb{X}$. Therefore, the global complexity of the algorithm is of order $O(mN(R/\delta)^d)$.

The main result of this paragraph is to provide an error analysis and rate of convergence for the approximation of $\bar{v}_i$ by $\hat{v}_i$.

Theorem 4.1 *There exists a constant K (not depending on h) such that*

$$\begin{aligned}
\big|\bar{v}_i(t_k, x) - \hat{v}_i(t_k, x)\big| \;\le\; & K \exp\big(K h^{-1} \|\vartheta - \hat{\vartheta}\|_2^2\big) \Big(1 + |x| + \frac{\delta}{h}\Big) \\
& \Big[\frac{\delta}{h} + h^{-1/2} \|\vartheta - \hat{\vartheta}\|_2 \Big(1 + |x| + \frac{\delta}{h}\Big) \\
& \quad + \frac{1}{Rh} \exp\big(K h^{-2} \|\vartheta - \hat{\vartheta}\|_4^4\big) \Big(1 + |x|^2 + (\frac{\delta}{h})^2\Big)\Big],
\end{aligned}$$

for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$. In the case where the switching costs c_{ij} do not depend on x, the above estimation is stengthened into:

$$\begin{aligned}
\big|\bar{v}_i(t_k, x) - \hat{v}_i(t_k, x)\big| \;\le\; & K\Big[h^{-1/2} \|\vartheta - \hat{\vartheta}\|_2 \exp\big(K h^{-1} \|\vartheta - \hat{\vartheta}\|_2^2\big) \Big(1 + |x| + \frac{\delta}{h}\Big) \\
& + \frac{\delta}{h} + \frac{1}{Rh} \exp\big(K h^{-2} \|\vartheta - \hat{\vartheta}\|_4^4\big) \Big(1 + |x|^2 + (\frac{\delta}{h})^2\Big)\Big].
\end{aligned}$$

Remark 4.1 The estimation in Theorem 4.1 consists of error terms related to

- the space discretization parameters δ, R, which have to be chosen s.t. δ/h and $1/Rh$ go to zero.
- the quantization error $\|\vartheta - \hat{\vartheta}\|_p$ of the normal distribution $\mathcal{N}(0, I_d)$, which converges to zero at a rate $N^{\frac{1}{d}}$, where N is the number of grid points chosen s.t. $h^{\frac{-1}{2}} N^{\frac{-1}{d}}$ goes to zero.

By combining with the discrete-time approximation error (3.14), and by choosing grid parameters δ, $1/R$ of order $h^{\frac{3}{2}}$, and a number of points N of order $1/h^d$, we see that the error estimate between the value function of the continuous-time optimal switching problem and its approximation by Markovian quantization is of order $h^{\frac{1}{2}}$. With these values of the parameters, we then see that the complexity of this Markovian quantization algorithm is of order $O(1/h^{4d+1})$.

Let us now focus on the proof of Theorem 4.1. First, notice from the dynamic programming principle that the value functions $\hat{v}_i$, $i \in \mathbb{I}_q$, admit the Markov control problem

representation:

$$\hat{v}_i(t_k,x) \;=\; \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\hat{X}^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell})h + g(\hat{X}^{t_k,x,\alpha}_{t_m}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\hat{X}^{t_k,x,\alpha}_{\tau_n}, \iota_{n-1}, \iota_n)\Big], \tag{4.4}$$

where $\hat{X}^{t_k,x,\alpha}$ is defined by

$$\hat{X}^{t_k,x,\alpha}_{t_k} \;=\; x, \qquad \hat{X}^{t_k,x,\alpha}_{t_{\ell+1}} \;=\; \mathrm{Proj}_{\mathbb{X}}\big(F^h_{I_{t_\ell}}(\hat{X}^{t_k,x,\alpha}_{t_\ell}, \hat{\vartheta}_{\ell+1})\big), \quad k\le \ell\le m-1,$$

for $\alpha\in\mathcal{A}^h_{t_k,i}$, and $\hat{\vartheta}_{k+1}$, $k=0,\ldots,m-1$, are iid, $\hat{\vartheta}$-distributed, and independent of $\mathcal{F}_{t_k}$. We first prove several estimates on $\hat{X}^{t_k,x,\alpha}$.

Lemma 4.1 *For each $p\ge 1$ there exists a constant K_p (not depending on h) such that*

$$\sup_{\alpha\in\mathcal{A}^h_{t_k,i},k\le\ell\le m}\Big\|\hat{X}^{t_k,x,\alpha}_{t_\ell}\Big\|_p + \sup_{\alpha\in\mathcal{A}^h_{t_k,i},k\le\ell\le m-1}\Big\|F^h_{I_{t_\ell}}(\hat{X}^{t_k,x,\alpha}_{t_\ell}, \hat{\vartheta}_{k+1})\Big\|_p \le K_p\exp\Big(K_p h^{-p/2}\|\vartheta-\hat{\vartheta}\|_p^p\Big)\Big(1+|x|+\frac{\delta}{h}\Big), \tag{4.5}$$

for all $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{X}\times\mathbb{I}_q$.

Proof. We fix $(t_k,x,i)\in\mathbb{T}_h\times\mathbb{X}\times\mathbb{I}_q$, $\alpha\in\mathcal{A}^h_{t_k,i}$, and denote $\hat{X}_{t_\ell}=\hat{X}^{t_k,x,\alpha}_{t_\ell}$, $k\le\ell\le m$. Denoting by $\mathbb{E}_l$ the conditional expectation w.r.t. $\mathcal{F}_{t_\ell}$, by a standard use of Gronwall's lemma and linear growth of b_i, σ_i, we have

$$\mathbb{E}_\ell\Big|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})\Big|^p \;\le\; e^{K_p h}\Big|\hat{X}_{t_\ell}\Big|^p + K_p h. \tag{4.6}$$

We will use the following convexity inequality : for a, $b\in\mathbb{R}_+$, $h\in[0,1]$,

$$(a+hb)^p \;\le\; (1+K_p h)a^p + K_p h b^p. \tag{4.7}$$

By definition of F^h, and the fact that $|\mathrm{Proj}_{\mathbb{X}}(y)|\le|y|+\delta$ for all $y\in\mathbb{R}^d$,

$$\begin{aligned}\Big|\hat{X}_{t_{\ell+1}}\Big| &\le \Big|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})\Big| + h^{1/2}\sigma_{I_{t_\ell}}(\hat{X}_{t_\ell})\big|\hat{\vartheta}_{\ell+1}-\vartheta_{\ell+1}\big| + \delta \\ &= \Big|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})\Big| + h\left(\frac{\sigma_{I_{t_\ell}}(\hat{X}_{t_\ell})\big|\hat{\vartheta}_{\ell+1}-\vartheta_{\ell+1}\big|}{h^{1/2}} + \frac{\delta}{h}\right)\end{aligned}$$

Combining this last inequality with (4.6), (4.7), linear growth of σ_i and the fact that $\hat{\vartheta}_{\ell+1},\vartheta_{\ell+1}$ are independent of $\mathcal{F}_{t_\ell}$, we obtain

$$\begin{aligned}\mathbb{E}_\ell\Big|\hat{X}_{t_{\ell+1}}\Big|^p &\le (1+K_p h)\big(e^{K_p h}|\hat{X}_{t_\ell}|^p + K_p h\big) + K_p h\left(\frac{\sigma_{I_{t_\ell}}(\hat{X}_{t_\ell})\|\vartheta-\hat{\vartheta}\|_p^p}{h^{p/2}} + \frac{\delta^p}{h^p}\right) \\ &\le \Big(1+K_p h + K_p h^{1-p/2}\|\vartheta-\hat{\vartheta}\|_p^p\Big)|\hat{X}_{t_\ell}|^p + K_p h\Big(1+\|\vartheta-\hat{\vartheta}\|_p^p h^{-p/2} + \frac{\delta^p}{h^p}\Big).\end{aligned}$$

By induction, taking the expectation, recalling that $h = \frac{T}{m}$, and since $\left(1+\frac{y}{m}\right)^m \le e^y$ for all $y \ge 0$, we obtain

$$
\begin{aligned}
\mathbb{E}\Big|\hat{X}_{t_{\ell+1}}\Big|^p &\le K_p \exp\Big(K_p h^{-p/2}\big\|\vartheta-\hat{\vartheta}\big\|_p^p\Big)\Big(1+|x|^p+\frac{\delta^p}{h^p}+h^{-p/2}\big\|\vartheta-\hat{\vartheta}\big\|_p^p\Big) \\
&\le K_p \exp\Big(K_p' h^{-p/2}\big\|\vartheta-\hat{\vartheta}\big\|_p^p\Big)\Big(1+|x|^p+\frac{\delta^p}{h^p}\Big),
\end{aligned}
$$

for all $k \le \ell \le m$. The estimate for $F^h(\hat{X}_{t_\ell}, \vartheta_{\ell+1})$ then follows from (4.6). $\square$

Lemma 4.2 *There exists some constant K (not depending on h) such that*

$$
\begin{aligned}
&\sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \Big\|\sup_{k\le\ell\le m}\big|\hat{X}^{t_k,x,\alpha}_{t_\ell}-\bar{X}^{t_k,x,\alpha}_{t_\ell}\big|\Big\|_2 \\
\le\;& K\Big[h^{-1/2}\big\|\vartheta-\hat{\vartheta}\big\|_2\exp\big(Kh^{-1/2}\big\|\vartheta-\hat{\vartheta}\big\|_2\big)\Big(1+|x|+\frac{\delta}{h}\Big) \\
&+\frac{\delta}{h}+\frac{1}{Rh}\exp\big(Kh^{-2}\big\|\vartheta-\hat{\vartheta}\big\|_4^4\big)\Big(1+|x|^2+\big(\frac{\delta}{h}\big)^2\Big)\Big],
\end{aligned}
\tag{4.8}
$$

for all $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$.

Proof. As before we fix (t_k, x, i), α and omit the dependence on (t_k, x, i, α) in $\hat{X}_{t_\ell}$. Let us first show an estimate on $\Big\|\hat{X}_{t_{\ell+1}} - \bar{X}_{t_{\ell+1}}\Big\|_2$. For $k \le \ell \le m-1$, we get

$$
\begin{aligned}
\Big\|\hat{X}_{t_{\ell+1}}-\bar{X}_{t_{\ell+1}}\Big\|_2 &\le \Big\|\hat{X}_{t_{\ell+1}}-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_2+\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})\Big\|_2 \\
&\quad+\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})-F^h_{I_{t_\ell}}(\bar{X}_{t_\ell},\vartheta_{\ell+1})\Big\|_2.
\end{aligned}
\tag{4.9}
$$

On the other hand, since

$$
\big|y-\mathrm{Proj}_{\mathbb{X}}(y)\big| \;\le\; \delta+|y|\mathbf{1}_{\{|y|\ge R\}} \;\le\; \delta+\frac{|y|^2}{R},
$$

by inequality (4.3), we have

$$
\Big\|\hat{X}_{t_{\ell+1}}-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_2 \;\le\; \delta+\frac{\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_4^2}{R}.
\tag{4.10}
$$

Furthermore by standard estimates for the Euler scheme (see e.g. Lemma A.1 in [16]), we have

$$
\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})-F^h_{I_{t_\ell}}(\bar{X}_{t_\ell},\vartheta_{\ell+1})\Big\|_2 \;\le\; (1+Kh)\Big\|\hat{X}_{t_\ell}-\bar{X}_{t_\ell}\Big\|_2,
$$

and by the linear growth property of σ and the fact that $\hat{\vartheta}_{\ell+1}, \vartheta_{\ell+1}$ are independent of $\mathcal{F}_{t_\ell}$,

$$
\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\vartheta_{\ell+1})-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_2 \;\le\; Kh^{1/2}\Big(1+\Big\|\hat{X}_{t_\ell}\Big\|_2\Big)\big\|\vartheta-\hat{\vartheta}\big\|_2.
\tag{4.11}
$$

Plugging these three inequalities into (4.9), we get :

$$
\begin{aligned}
\Big\|\hat{X}_{t_{\ell+1}}-\bar{X}_{t_{\ell+1}}\Big\|_2 &\le (1+Kh)\Big\|\hat{X}_{t_\ell}-\bar{X}_{t_\ell}\Big\|_2+Kh^{1/2}\Big(\Big\|\hat{X}_{t_\ell}\Big\|_2+1\Big)\big\|\vartheta-\hat{\vartheta}\big\|_2 \\
&\quad+\delta\;+\;\frac{\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_4^2}{R}.
\end{aligned}
$$

Finally since $\hat{X}_{t_k} = \bar{X}_{t_k} = x$, we obtain by induction, and using the estimates (4.5) on $\Big\|F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_4$:

$$\begin{aligned}\Big\|\hat{X}_{t_\ell}-\bar{X}_{t_\ell}\Big\|_2 &\leq K\Big[h^{-1/2}\big\|\vartheta-\hat{\vartheta}\big\|_2\exp\big(Kh^{-1}\big\|\vartheta-\hat{\vartheta}\big\|_2^2\big)\Big(1+|x|+\frac{\delta}{h}\Big)+\frac{\delta}{h}\\ &\quad+\frac{1}{Rh}\exp\big(Kh^{-2}\big\|\vartheta-\hat{\vartheta}\big\|_4^4\big)\Big(1+|x|^2+(\frac{\delta}{h})^2\Big)\Big],\end{aligned}\tag{4.12}$$

for all $k \leq \ell \leq m$. Now by definition of $\hat{X}_{t_k}$, $\bar{X}_{t_k}$, we may write for $k \leq \ell \leq m-1$:

$$\begin{aligned}\hat{X}_{t_{\ell+1}}-\bar{X}_{t_{\ell+1}} &= (\hat{X}_{t_\ell}-\bar{X}_{t_\ell})+h\big(b(\hat{X}_{t_\ell},I_{t_\ell})-b(\bar{X}_{t_\ell},I_{t_\ell})\big)\\ &\quad+\sqrt{h}\big(\sigma(\hat{X}_{t_\ell},I_{t_\ell})\hat{\vartheta}_{\ell+1}-\sigma(\bar{X}_{t_\ell},I_{t_\ell})\vartheta_{\ell+1}\big)\\ &\quad+\mathrm{Proj}_{\mathbb{X}}\big(F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\big)-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1}),\end{aligned}$$

Since $\hat{X}_{t_k} = \bar{X}_{t_k}$ $(=x)$, we obtain by induction:

$$\begin{aligned}\Big\|\sup_{k\leq\ell\leq m}\Big|\hat{X}_{t_\ell}-\bar{X}_{t_\ell}\Big|\Big\|_2 &\leq h\sum_{\ell=k}^{m-1}\Big\|b(\hat{X}_{t_\ell},I_{t_\ell})-b(\bar{X}_{t_\ell},I_{t_\ell})\Big\|_2\\ &\quad+\sqrt{h}\Big\|\sup_{k\leq\ell\leq m}\big|\sum_{r\leq\ell}\sigma(\hat{X}_{t_r},I_{t_r})\hat{\vartheta}_{r+1}-\sigma(\bar{X}_{t_r},I_{t_r})\vartheta_{r+1}\big|\Big\|_2\\ &\quad+\sum_{\ell=k}^{m-1}\Big\|\mathrm{Proj}_{\mathbb{X}}\big(F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\big)-F^h_{I_{t_\ell}}(\hat{X}_{t_\ell},\hat{\vartheta}_{\ell+1})\Big\|_2.\end{aligned}\tag{4.13}$$

We now bound each of the three terms in the right hand side of (4.13). First, by the Lipschitz property of b and (4.12), we have

$$\begin{aligned}&h\sum_{\ell=k}^{m-1}\big\|b(\hat{X}_{t_\ell},I_{t_\ell})-b(\bar{X}_{t_\ell},I_{t_\ell})\big\|_2\\ \leq\ & K\Big[h^{-1/2}\big\|\vartheta-\hat{\vartheta}\big\|_2\exp\big(Kh^{-1}\big\|\vartheta-\hat{\vartheta}\big\|_2^2\big)\Big(1+|x|+\frac{\delta}{h}\Big)\\ &\quad+\frac{\delta}{h}+\frac{1}{Rh}\exp\big(Kh^{-2}\big\|\vartheta-\hat{\vartheta}\big\|_4^4\big)\Big(1+|x|^2+(\frac{\delta}{h})^2\Big)\Big].\end{aligned}$$

Next, recalling that $\hat{\vartheta}_{\ell+1}$ is independent of $\mathcal{F}_{t_\ell}$, with distribution law $\hat{\vartheta}$, and since $\hat{\vartheta}$ is an optimal L^2-quantizer of ϑ, it follows that $\mathbb{E}[\hat{\vartheta}_{\ell+1}|\mathcal{F}_{t_\ell}] = \mathbb{E}[\hat{\vartheta}] = \mathbb{E}[\vartheta] = 0$. Thus, the process $(\sum_{r\leq\ell}\sigma(\hat{X}_{t_r},I_{t_r})\hat{\vartheta}_{r+1}-\sigma(\bar{X}_{t_r},I_{t_r})\vartheta_{r+1})_\ell$ is a $\mathcal{F}_{t_\ell}$-martingale, and from Doob's inequality, we have:

$$\begin{aligned}&\Big\|\sup_{k\leq\ell\leq m}\big|\sum_{r\leq\ell}\sigma(\hat{X}_{t_r},I_{t_r})\hat{\vartheta}_{r+1}-\sigma(\bar{X}_{t_r},I_{t_r})\vartheta_{r+1}\big|\Big\|_2\\ \leq\ & K\Big(\mathbb{E}\Big[\sum_{\ell=k}^{m-1}\big|\sigma(\hat{X}_{t_\ell},I_{t_\ell})\hat{\vartheta}_{\ell+1}-\sigma(\bar{X}_{t_\ell},I_{t_\ell})\vartheta_{\ell+1}\big|^2\Big]\Big)^{\frac{1}{2}}.\end{aligned}$$

By writing from the Lipschitz condition on σ_i that

$$\begin{aligned}\big|\sigma(\hat{X}_{t_\ell},I_{t_\ell})\hat{\vartheta}_{\ell+1}-\sigma(\bar{X}_{t_\ell},I_{t_\ell})\vartheta_{\ell+1}\big|^2 &\leq K\Big(\big|\hat{X}_{t_\ell}-\bar{X}_{t_\ell}\big|^2\big|\vartheta_{\ell+1}\big|^2\\ &\quad+\big(1+\big|\hat{X}_{t_\ell}\big|^2\big)\big|\vartheta_{\ell+1}-\hat{\vartheta}_{\ell+1}\big|^2\Big),\end{aligned}$$

and since $\vartheta_{\ell+1}, \hat{\vartheta}_{\ell+1}$ are independent of $\mathcal{F}_{t_\ell}$, we then obtain

$$
\begin{aligned}
& \sqrt{h}\Big\| \sup_{k\leq \ell\leq m} \big|\sum_{r\leq \ell} \sigma(\hat{X}_{t_r}, I_{t_r})\hat{\vartheta}_{r+1} - \sigma(\bar{X}_{t_r}, I_{t_r})\vartheta_{r+1}\big|\Big\|_2 \\
\leq\ & K \sup_{k\leq \ell\leq m-1} \Big[\big\|\hat{X}_{t_\ell} - \bar{X}_{t_\ell}\big\|_2 + \big(1 + \|\hat{X}_{t_\ell}\|_2\big)\|\vartheta - \hat{\vartheta}\|_2\Big] \\
\leq\ & K\Big[h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2 \exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1 + |x| + \frac{\delta}{h}\Big) \\
& \quad + \frac{\delta}{h} + \frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1 + |x|^2 + \big(\frac{\delta}{h}\big)^2\Big)\Big],
\end{aligned}
$$

where we used the estimates (4.5) and (4.12). Finally the third term in (4.13) is bounded as before by (4.10). □

Proof of Theorem 4.1. For $(t_k, x, i) \in \mathbb{T}_h \times \mathbb{X} \times \mathbb{I}_q$, we show as in the proof of Theorem 3.2 that we can restrict to strategies $\alpha \in \mathcal{A}^h_{t_k,i}$ such that

$$
\mathbb{E}\big|N(\alpha)\big|^2 \ \leq\ K\Big(1 + \sup_{k\leq \ell\leq m} \Big\|\hat{X}^{t_k,x,\alpha}_{t_\ell}\Big\|_2^2\Big),
$$

for some constant K, not depending on (t_k, x, i, h). By using the estimation (4.5), this means that the supremum in the representation (3.1) of $\hat{v}_i(t_k, x)$ can be taken over the subset

$$
\hat{\mathcal{A}}^{h,K}_{t_k,i}(x) \ = \ \Big\{\alpha \in \mathcal{A}^h_{t_k,i} \text{ s.t. } \mathbb{E}|N(\alpha)|^2 \leq K\exp\Big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\Big)\Big(1 + |x|^2 + \frac{\delta^2}{h^2}\Big)\Big\}.
$$

Then, for $\alpha \in \hat{\mathcal{A}}^{h,K}_{t_k,i}(x)$, we have under **(Hl)** and by Cauchy-Schwarz inequality

$$
\begin{aligned}
& \mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(\bar{X}^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell}) - f(\hat{X}^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell})\big| + \big|g(\bar{X}^{t_k,x,\alpha}_{t_m}, I_{t_m}) - g(\hat{X}^{t_k,x,\alpha}_{t_m}, I_{t_m})\big| \\
& \quad + \sum_{n=1}^{N(\alpha)} \big|c(\bar{X}^{t_k,x,\alpha}_{\tau_n}, \iota_{n-1}, \iota_n) - c(\hat{X}^{h,t_k,x,\alpha}_{\tau_n}, \iota_{n-1}, \iota_n)\big|\Big] \\
\leq\ & K\mathbb{E}\Big[(1 + N(\alpha))\big(\sup_{k\leq \ell\leq m} \big|\bar{X}^{t_k,x,\alpha}_{t_\ell} - \hat{X}^{t_k,x,\alpha}_{t_\ell}\big|\big)\Big] \\
\leq\ & K\exp\Big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\Big)\ \big(1 + |x| + \frac{\delta}{h}\big)\Big\|\sup_{k\leq \ell\leq m} \big|\bar{X}^{t_k,x,\alpha}_{t_\ell} - \hat{X}^{t_k,x,\alpha}_{t_\ell}\big|\Big\|_2 \\
\leq\ & K\exp\big(Kh^{-1}\|\vartheta - \hat{\vartheta}\|_2^2\big)\Big(1 + |x| + \frac{\delta}{h}\Big)\Big[\frac{\delta}{h} + h^{-1/2}\|\vartheta - \hat{\vartheta}\|_2\Big(1 + |x| + \frac{\delta}{h}\Big) \\
& \quad + \frac{1}{Rh}\exp\big(Kh^{-2}\|\vartheta - \hat{\vartheta}\|_4^4\big)\Big(1 + |x|^2 + \big(\frac{\delta}{h}\big)^2\Big)\Big], \qquad (4.14)
\end{aligned}
$$

by Lemma 4.2. Taking the supremum over $\alpha \in \hat{\mathcal{A}}^{h,K}_{t_k,i}(x)$ in the above inequality, we obtain an estimate for $|\bar{v}_i(t_k, x) - \hat{v}_i(t_k, x)|$ with an upper bound given by the r.h.s. of (4.14), which gives the required result.

Finally, notice that in the special case where the switching cost functions c_{ij} do not depend on x, we have

$$
\begin{aligned}
\big|\bar v_i(t_k,x) - \hat v_i(t_k,x)\big| &\leq \sup_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(\bar X^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell}) - f(\hat X^{t_k,x,\alpha}_{t_\ell}, I_{t_\ell})\big| \\
&\qquad + \big|g(\bar X^{t_k,x,\alpha}_{t_m}, I_{t_m}) - g(\hat X^{t_k,x,\alpha}_{t_m}, I_{t_m})\big|\Big] \\
&\leq K \sup_{\alpha\in\mathcal{A}^h_{t_k,i}, k\leq\ell\leq m} \mathbb{E}\big|\bar X^{t_k,x,\alpha}_{t_\ell} - \hat X^{t_k,x,\alpha}_{t_\ell}\big| \\
&\leq K\Big[h^{-1/2}\big\|\vartheta-\hat\vartheta\big\|_2 \exp\big(Kh^{-1}\big\|\vartheta-\hat\vartheta\big\|_2^2\big)\Big(1+|x|+\frac{\delta}{h}\Big) \\
&\qquad + \frac{\delta}{h} + \frac{1}{Rh}\exp\big(Kh^{-2}\big\|\vartheta-\hat\vartheta\big\|_4^4\big)\Big(1+|x|^2+\big(\frac{\delta}{h}\big)^2\Big)\Big],
\end{aligned}
$$

by the estimate in Lemma 4.2. □

4.2 Marginal quantization in the uncontrolled diffusion case

In this paragraph, we consider the special case where the diffusion X is not controlled, i.e. $b_i = b$, $\sigma_i = \sigma$. The Euler scheme for X, denoted by $\bar X$, is given by:

$$
\begin{aligned}
\bar X_0 \;=\; X_0, \quad \bar X_{t_{k+1}} &= F^h(\bar X_{t_k}, \vartheta_{k+1}) \\
&:= \bar X_{t_k} + b(\bar X_{t_k})h + \sigma(\bar X_{t_k})\sqrt{h}\,\vartheta_{k+1}, \quad k=0,\ldots,m-1,
\end{aligned}
$$

where $\vartheta_{k+1} = (W_{t_{k+1}} - W_{t_k})/\sqrt{h}$, $k=0,\ldots,m-1$, are iid, $\mathcal{N}(0,I_d)$-distributed, independent of $\mathcal{F}_{t_k}$. Let us recall the well-known estimate: for any $p \geq 1$, there exists some K_p s.t.

$$
\big\|\bar X_{t_k}\big\|_p \;\leq\; K_p(1+\big\|X_0\big\|_p). \tag{4.15}
$$

Notice that the backward dynamic programming formulae (4.1)-(4.2) for $\bar v_i$ can be written in this case as:

$$
\begin{aligned}
\bar v_i(t_m,.) &= g_i(.), \quad i\in\mathbb{I}_q \\
\bar v_i(t_k,.) &= \max_{j\in\mathbb{I}_q}[P^h\bar v_j(t_{k+1},.) + hf_j - c_{ij}].
\end{aligned} \tag{4.16}
$$

Here P^h is the probability transition kernel of the Markov chain $\bar X$, given by:

$$
P^h\varphi(x) \;=\; \mathbb{E}\big[\varphi(\bar X_{t_{k+1}})|\bar X_{t_k}=x\big] \;=\; \mathbb{E}[\varphi(F^h(x,\vartheta))], \tag{4.17}
$$

where ϑ is $\mathcal{N}(0,I_d)$-distributed. Let us next consider the family of discrete-time processes $(\bar Y^i_{t_k})_{k=0,\ldots,m}$, $i\in\mathbb{I}_q$, defined by:

$$
\bar Y^i_{t_k} \;=\; \bar v_i(t_k,\bar X_{t_k}), \quad k=0,\ldots,m,\ i\in\mathbb{I}_q.
$$

Remark 4.2 By the Markov property of the Euler scheme $\bar X$ w.r.t. $(\mathcal{F}_{t_k})_k$, we see that $(\bar Y^i_{t_k})_{k=0,\ldots,m}$, $i\in\mathbb{I}_q$, satisfy the backward induction:

$$
\begin{aligned}
\bar Y^i_{t_m} &= g_i(\bar X_{t_m}) \;=\; g_i(\bar X_T), \quad i\in\mathbb{I}_q \\
\bar Y^i_{t_k} &= \max_{j\in\mathbb{I}_q}\Big\{\mathbb{E}\big[\bar Y^j_{t_{k+1}}\big|\mathcal{F}_{t_k}\big] + hf_j(\bar X_{t_k}) - c_{ij}(\bar X_{t_k})\Big\}, \quad k=0,\ldots,m-1,
\end{aligned}
$$

and is represented as

$$\bar{Y}^i_{t_k} = \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}^h_{t_k,i}} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\bar{X}_{t_\ell}, I_{t_\ell})h + g(\bar{X}_{t_m}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\bar{X}_{\tau_n}, \iota_{n-1}, \iota_n)\Big|\mathcal{F}_{t_k}\Big].$$

On the other hand, the continuous-time optimal switching problem (2.4) admits a representation in terms of the following reflected Backward Stochastic Differential Equations (BSDE):

$$\begin{aligned} Y^i_t &= g_i(X_T) + \int_t^T f(X_s)ds - \int_t^T Z^i_s dW_s + K^i_T - K^i_t, \quad i \in \mathbb{I}_q,\ 0 \le t \le T, \\ Y^i_t &\ge \max_{j\neq i}[Y^j_t - c_{ij}(X_t)] \quad \text{and} \quad \int_0^T \big(Y^i_t - \max_{j\neq i}[Y^j_t - c_{ij}(X_t)]\big)dK^i_t \;=\; 0. \end{aligned} \tag{4.18}$$

We know from [6], [10] or [9] that there exists a unique solution $(Y, Z, K) = (Y^i, Z^i, K^i)_{i\in\mathbb{I}_q}$ solution to (4.18) with $Y \in \mathcal{S}^2(\mathbb{R}^q)$, the set of adapted continuous processes valued in $\mathbb{R}^q$ s.t. $\mathbb{E}[\sup_{0\le t\le T}|Y_t|^2] < \infty$, $Z \in \mathcal{M}^2(\mathbb{R}^q)$, the set of predictable processes valued in $\mathbb{R}^q$ s.t. $\mathbb{E}[\int_0^T |Z_t|^2 dt] < \infty$, and $K^i \in \mathcal{S}^2(\mathbb{R})$, $K^i_0 = 0$, K^i is nondecreasing. Moreover, we have

$$\begin{aligned} Y^i_t &= v_i(t, X_t), \quad i \in \mathbb{I}_q, \\ &= \operatorname*{ess\,sup}_{\alpha\in\mathcal{A}_{t,i}} \mathbb{E}\Big[\int_t^T f(X_s, I_s)ds + g(X_T, I_T) - \sum_{n=1}^{N(\alpha)} c(X_{\tau_n}, \iota_{n-1}, \iota_n)\Big|\mathcal{F}_t\Big], \ \ 0 \le t \le T. \end{aligned}$$

We propose now an optimal quantization method in the vein of [1] for optimal stopping problems, for a computational approximation of $(\bar{Y}^i_{t_k})_{k=0,\dots,m}$. This is based on results about optimal quantization of each marginal distribution of the Markov chain $(\bar{X}_{t_k})_{0\le k\le m}$. Let us recall the construction. For each time step $k = 0, \dots, m$, we are given a grid $\Gamma_k = \{x^1_k, \dots, x^{N_k}_k\}$ of N_k points in $\mathbb{R}^d$, and we define the quantizer $\hat{X}_k = \mathrm{Proj}_k(\bar{X}_{t_k})$ of $\bar{X}_{t_k}$ where Proj_k denotes a closest neighbour projection on Γ_k. For N_k being fixed, the grid Γ_k is said to be L^p-optimal if it minimizes the L^p-quantization error: $\|\bar{X}_{t_k} - \mathrm{Proj}_k(\bar{X}_{t_k})\|_p$. Optimal grids Γ_k are produced by a stochastic recursive algorithm, called Competitive Learning Vector Quantization (or also Kohonen Algorithm), and relying on Monte-Carlo simulations of $\bar{X}_{t_k}$, $k = 0, \dots, m$. We refer to [15] for details about the CLVQ algorithm. We also compute the transition weights

$$\pi^{ll'}_k = \mathbb{P}[\hat{X}_{k+1} = x^{l'}_{k+1}|\hat{X}_k = x^l_k] \;=\; \frac{\mathbb{P}\big[(\bar{X}_{t_{k+1}}, \bar{X}_{t_k}) \in C_{l'}(\Gamma_{k+1}) \times C_l(\Gamma_k)\big]}{\mathbb{P}\big[\bar{X}_{t_k} \in C_l(\Gamma_k)\big]},$$

where $C_l(\Gamma_k) \subset \{x \in \mathbb{R}^d : |x - x^l_k| = \min_{y\in\Gamma_k}|x - y|\}$, $l = 1, \dots, N_k$, is a Voronoi tesselation of Γ_k. These weights can be computed either during the CLVQ phase, or by a regular Monte-Carlo simulation once the grids Γ_k are settled. The associated discrete probability transition $\hat{P}_k$ from $\hat{X}_k$ to $\hat{X}_{k+1}$, $k = 0, \dots, m-1$, is given by:

$$\hat{P}_k\varphi(x^l_k) \;:=\; \sum_{l'=1}^{N_{k+1}} \pi^{ll'}_k \varphi(x^{l'}_{k+1}) \;=\; \mathbb{E}\big[\varphi(\hat{X}_{k+1})\big|\hat{X}_k = x^l_k\big].$$

One then defines by backward induction the sequence of $\mathbb{R}^q$-valued functions $\hat{v}_k = (\hat{v}_k^i)_{i\in\mathbb{I}_q}$ computed explicitly on Γ_k, $k = 0,\ldots,m$, by the quantization tree algorithm:

$$\begin{aligned} \hat{v}_m^i &= g_i, \quad i \in \mathbb{I}_q, \\ \hat{v}_k^i &= \max_{j\in\mathbb{I}_q} \big[\hat{P}_k \hat{v}_{k+1}^j + hf_j - c_{ij}\big], \quad k = 0,\ldots,m-1. \end{aligned} \tag{4.19}$$

The discrete-time processes $(\bar{Y}_{t_k}^i)_{k=0,\ldots,m}$, $i \in \mathbb{I}_q$, are then approximated by the quantized processes $(\hat{Y}_k^i)_{k=0,\ldots,m}$, $i \in \mathbb{I}_q$ defined by

$$\hat{Y}_k^i = \hat{v}_k^i(\hat{X}_k), \quad k = 0,\ldots,m,\ i \in \mathbb{I}_q.$$

The rest of this section is devoted to the error analysis between $\bar{Y}^i$ and $\hat{Y}^i$. The analysis follows arguments as in [2] for optimal stopping problems, but has to be slightly modified since the functions $\bar{v}_i(t_k,.)$ are not Lipschitz in general when the switching costs depend on x. Let us introduce the subset $LLip(\mathbb{R}^d)$ of measurable functions φ on $\mathbb{R}^d$ satisfying:

$$|\varphi(x) - \varphi(y)| \leq K(1 + |x| + |y|)|x - y|, \quad \forall x, y \in \mathbb{R}^d,$$

for some positive constant K, and denote by

$$[\varphi]_{LLip} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|\varphi(x) - \varphi(y)|}{(1 + |x| + |y|)|x - y|}.$$

Lemma 4.3 *The functions $\bar{v}_i(t_k,.)$, $k = 0,\ldots,m$, $i \in \mathbb{I}_q$, lie in $LLip(\mathbb{R}^d)$, and $[\bar{v}_i(t_k,.)]_{LLip}$ is bounded by a constant not depending on (k,i,h).*

Proof. We set $\bar{v}_k^i = \bar{v}_i(t_k,.)$. From the representation (3.12), we have

$$\bar{v}_k^i(x) = \sup_{\alpha\in\mathcal{A}_{t_k,i}^h} \mathbb{E}\Big[\sum_{\ell=k}^{m-1} f(\bar{X}_{t_\ell}^{t_k,x}, I_{t_\ell})h + g(\bar{X}_{t_m}^{t_k,x}, I_{t_m}) - \sum_{n=1}^{N(\alpha)} c(\bar{X}_{\tau_n}^{t_k,x}, \iota_{n-1}, \iota_n)\Big],$$

where $\bar{X}^{t_k,x}$ is the solution to the Euler scheme starting from x at time t_k. From (4.15), we see, as in the proof of Theorem 3.2, that in the above representation for $\bar{v}_k^i(x)$, one can restrict the supremum to $\mathcal{A}_{t_k,i}^{h,K}(x) = \big\{\alpha \in \mathcal{A}_{t_k,i}^h \text{ s.t. } \mathbb{E}|N(\alpha)|^2 \leq K(1 + |x|^2)\big\}$ for some positive constant K not depending on (t_k,x,i,h). Then, as in the proof of Theorem 4.1, we have for any $x,y \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}_{t_k,i}^{h,K}(x) \cup \mathcal{A}_{t_k,i}^{h,K}(y)$,

$$\begin{aligned} &\mathbb{E}\Big[\sum_{\ell=k}^{m-1} h\big|f(\bar{X}_{t_\ell}^{t_k,x}, I_{t_\ell}) - f(\bar{X}_{t_\ell}^{t_k,y}, I_{t_\ell})\big| + \big|g(\bar{X}_{t_m}^{t_k,x}, I_{t_m}) - g(\bar{X}_{t_m}^{t_k,y}, I_{t_m})\big| \\ &\qquad + \sum_{n=1}^{N(\alpha)} \big|c(\bar{X}_{\tau_n}^{t_k,x}, \iota_{n-1}, \iota_n) - c(\bar{X}_{\tau_n}^{t_k,x}, \iota_{n-1}, \iota_n)\big|\Big] \\ &\leq K\big(1 + \big\|N(\alpha)\big\|_2\big)\Big\| \sup_{k\leq\ell\leq m} \big|\bar{X}_{t_\ell}^{t_k,x} - \bar{X}_{t_\ell}^{t_k,y}\big|\Big\|_2 \\ &\leq K(1 + |x| + |y|)|x - y|, \end{aligned}$$

by standard Lipschitz estimates on the Euler scheme. By taking the supremum over $\mathcal{A}^{h,K}_{t_k,i}(x)$ $\cup\ \mathcal{A}^{h,K}_{t_k,i}(y)$ in the above inequality, this shows that

$$|\bar{v}^i_k(x) - \bar{v}^i_k(y)| \quad \leq \quad K(1 + |x| + |y|)|x - y|,$$

i.e. $\bar{v}^i_k \in LLip(\mathbb{R}^d)$ with $[\bar{v}^i_k]_{LLip} \leq K$. □

The next Lemma shows that the probability transition kernel of the Euler scheme preserves the growth linear Lipschitz property.

Lemma 4.4 *For any $\varphi \in LLip(\mathbb{R}^d)$, the function $P^h\varphi$ also lies in $LLip(\mathbb{R}^d)$, and there exists some constant K, not depending on h, such that*

$$[P^h\varphi]_{LLip} \quad \leq \quad \sqrt{3}(1 + O(h))[\varphi]_{LLip},$$

where $O(h)$ denotes any function s.t. $O(h)/h$ is bounded when h goes to zero.

Proof. From (4.17) and Cauchy-Schwarz inequality, we have for any $x, y \in \mathbb{R}^d$:

$$\begin{aligned}
& |P^h\varphi(x) - P^h\varphi(y)| \\
\leq \;& \Big(\mathbb{E}\big|\varphi(F^h(x,\vartheta)) - \varphi(F^h(y,\vartheta))\big|^2\Big)^{1/2} \\
\leq \;& [\varphi]_{LLip}\Big(\mathbb{E}\big|\big(1 + |F^h(x,\vartheta)| + |F^h(y,\vartheta)|\big)^2\big|F^h(x,\vartheta) - F^h(y,\vartheta)\big|^2\Big)^{1/2} \\
\leq \;& \sqrt{3}[\varphi]_{LLip}\Big(\mathbb{E}\big[(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2\big]\Big)^{\frac{1}{2}}, \quad (4.20)
\end{aligned}$$

where we used the relation $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Since ϑ has a symmetric distribution, we have

$$\begin{aligned}
& \mathbb{E}\Big[\big(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2\big)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2\Big] \\
= \;& \frac{1}{2}\mathbb{E}\Big[\big(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2\big)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2 \\
& \quad + \big(1 + |F^h(x,-\vartheta)|^2 + |F^h(y,-\vartheta)|^2\big)|F^h(x,-\vartheta) - F^h(y,-\vartheta)|^2\Big]
\end{aligned}$$

A straightforward calculation gives

$$\begin{aligned}
& \frac{1}{2}\Big[\big(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2\big)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2 \\
& \quad + \big(1 + |F^h(x,-\vartheta)|^2 + |F^h(y,-\vartheta)|^2\big)|F^h(x,-\vartheta) - F^h(y,-\vartheta)|^2\Big] \\
= \;& \big(1 + |x + hb(x)|^2 + |y + hb(y)|^2 + h|\sigma(x)\vartheta|^2 + h|\sigma(y)\vartheta|^2\big)\big|x - y + h(b(x) - b(y))\big|^2 \\
& + h|(\sigma(x) - \sigma(y))\vartheta|^2\big(|x + hb(x)|^2 + |y + hb(y)|^2\big) \\
& + 4h\Big[\big(x + hb(x)|\sigma(x)\vartheta\big) + \big(y + hb(y)|\sigma(y)\vartheta\big)\Big]\big(x - y + h(b(x) - b(y))|(\sigma(x) - \sigma(y))\vartheta\big) \\
& + h^2(|\sigma(x)\vartheta|^2 + |\sigma(y)\vartheta|^2)|(\sigma(x) - \sigma(y))\vartheta|^2.
\end{aligned}$$

By Lipschitz continuity of b and σ, and the fact that $\mathbb{E}|\vartheta|^4 < \infty$, we deduce that

$$\begin{aligned}
& \mathbb{E}\Big[(1 + |F^h(x,\vartheta)|^2 + |F^h(y,\vartheta)|^2)|F^h(x,\vartheta) - F^h(y,\vartheta)|^2\Big] \\
\leq \;& (1 + O(h))(1 + |x|^2 + |y|^2)|x - y|^2.
\end{aligned}$$

Plugging this last inequality into (4.20) shows the required result. $\square$

We now pass to the main result of this section by providing some a priori estimates for $\|\bar{Y}_{t_k} - \hat{Y}_k\|$ in terms of the quantization error $\|\bar{X}_{t_k} - \hat{X}_k\|$.

Theorem 4.2 *There exists some positive constant K, not depending on h, such that*

$$\max_{i\in\mathbb{I}_q} \big\|\bar{Y}^i_{t_k} - \hat{Y}^i_k\big\|_p \quad \leq \quad K\sum_{\ell=k}^{m}(1 + \|X_0\|_r + \|\hat{X}_\ell\|_r)\big\|\bar{X}_{t_\ell} - \hat{X}_\ell\big\|_s, \tag{4.21}$$

for any $k = 0, \ldots, m$, and $(p, r, s) \in (1, \infty)$ s.t. $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.

Proof. We set $\bar{v}^i_k = \bar{v}_i(t_k, .)$, and by misuse of notations, we also set $\bar{Y}^i_k = \bar{Y}^i_{t_k} = \bar{v}^i_k(\bar{X}_k)$. From the recursive induction (4.16) (resp. (4.19)) on $\bar{v}^i_k$ (resp. $\hat{v}^i_k$), and the trivial inequality $|\max_j \bar{a}_j - \max_j \hat{a}_j| \leq \max_j |\bar{a}_j - \hat{a}_j|$, we have for all $i \in \mathbb{I}_q$:

$$\begin{aligned}
|\bar{Y}^i_k - \hat{Y}^i_k| \quad &= \quad |\bar{v}^i_k(\bar{X}_{t_k}) - \hat{v}^i_k(\hat{X}_k)| \\
&\leq \quad \max_{j\in\mathbb{I}_q}\big|\big[P^h\bar{v}^j_{k+1}(\bar{X}_{t_k}) + hf_j(\bar{X}_{t_k}) - c_{ij}(\bar{X}_{t_k})\big] - \big[\hat{P}_k\hat{v}^j_{k+1}(\hat{X}_k) + hf_j(\hat{X}_k) - c_{ij}(\hat{X}_k)\big]\big| \\
&\leq \quad \max_{j\in\mathbb{I}_q}\Big[\big|P^h\bar{v}^j_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^j_{k+1}(\hat{X}_k)\big| + h\big|f_j(\bar{X}_{t_k}) - f_j(\hat{X}_k)\big| + \big|c_{ij}(\bar{X}_{t_k}) - c_{ij}(\hat{X}_k)\big|\Big] \\
&\leq \quad K\big|\bar{X}_{t_k} - \hat{X}_k\big| + \max_{j\in\mathbb{I}_q}\big|P^h\bar{v}^j_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^j_{k+1}(\hat{X}_k)\big|
\end{aligned}$$

by the Lipschitz property of f_j and c_{ij}, and so

$$\max_{i\in\mathbb{I}_q}\Big\|\bar{Y}^i_k - \hat{Y}^i_k\Big\|_p \quad \leq \quad K\Big\|\bar{X}_{t_k} - \hat{X}_k\Big\|_p + \max_{i\in\mathbb{I}_q}\Big\|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^i_{k+1}(\hat{X}_k)\Big\|_p \tag{4.22}$$

Writing $\hat{\mathbb{E}}_k$ for the conditional expectation w.r.t. $\hat{X}_k$, we have for any $i \in \mathbb{I}_q$

$$\begin{aligned}
&\big|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^i_{k+1}(\hat{X}_k)\big| \\
\leq \quad &\big|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - P^h\bar{v}^i_{k+1}(\hat{X}_k)\big| + \big|P^h\bar{v}^i_{k+1}(\hat{X}_k) - \hat{\mathbb{E}}_k[P^h v^i_{k+1}(\bar{X}_{t_k})]\big| \\
&\quad + \big|\hat{\mathbb{E}}_k[P^h\bar{v}^i_{k+1}(\bar{X}_{t_k})] - \hat{P}_k\hat{v}^i_{k+1}(\hat{X}_k)\big| \\
= \quad &\big|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - P^h\bar{v}^i_{k+1}(\hat{X}_k)\big| + \big|\hat{\mathbb{E}}_k[P^h\bar{v}^i_{k+1}(\hat{X}_k) - P^h\bar{v}^i_{k+1}(\bar{X}_{t_k})]\big| \\
&\quad + \big|\hat{\mathbb{E}}_k[\bar{Y}^i_{k+1} - \hat{Y}^i_{k+1}]\big|.
\end{aligned}$$

Since $\hat{\mathbb{E}}_k$ is a L^p-contraction, we then obtain

$$\begin{aligned}
&\Big\|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - \hat{P}_k\hat{v}^i_{k+1}(\hat{X}_k)\Big\|_p \\
\leq \quad &2\Big\|P^h\bar{v}^i_{k+1}(\bar{X}_{t_k}) - P^h\bar{v}^i_{k+1}(\hat{X}_k)\Big\|_p + \Big\|\bar{Y}^i_{k+1} - \hat{Y}^i_{k+1}\Big\|_p \\
\leq \quad &K(1 + O(h))\Big\|\big(1 + |\bar{X}_{t_k}| + |\hat{X}_k|\big)|\bar{X}_{t_k} - \hat{X}_k|\Big\|_p + \Big\|\bar{Y}^i_{k+1} - \hat{Y}^i_{k+1}\Big\|_p \\
\leq \quad &K(1 + O(h))\big(1 + \|X_0\|_r + \|\hat{X}_k\|_r\big)\Big\|\bar{X}_{t_k} - \hat{X}_k\Big\|_s + \Big\|\bar{Y}^i_{k+1} - \hat{Y}^i_{k+1}\Big\|_p,
\end{aligned} \tag{4.23}$$

where we used Lemmata 4.4 and 4.3, Hölder's inequality and (4.15). Substituting (4.23) into (4.22), we get

$$
\begin{aligned}
& \max_{i\in\mathbb{I}_q} \left\| \bar{Y}_k^i - \hat{Y}_k^i \right\|_p \\
\leq \quad & K(1+O(h))\Big(1 + \left\|X_0\right\|_r + \left\|\hat{X}_k\right\|_r\Big) \left\|\bar{X}_{t_k} - \hat{X}_k\right\|_s + \max_{i\in\mathbb{I}_q} \left\|\bar{Y}_{k+1}^i - \hat{Y}_{k+1}^i\right\|_p,
\end{aligned}
$$

for all $k = 0,\dots,m-1$. Since $\max_{i\in\mathbb{I}_q} \left\|\bar{Y}_m^i - \hat{Y}_m^i\right\|_p = \max_{i\in\mathbb{I}_q} \left\|g_i(\bar{X}_{t_m}) - g(\hat{X}_m)\right\|_p \leq K\left\|\bar{X}_{t_m} - \hat{X}_m\right\|_p$ by the Lipschitz condition on g_i, we conclude by induction. □

Remark 4.3 Assume that $\hat{X}_k$ is chosen to be an L^2-optimal quantizer of $\bar{X}_{t_k}$ for each $k = 0,\dots,m$. It is in particular a stationary quantizer in the sense that $\mathbb{E}[\bar{X}_{t_k}|\hat{X}_k] = \hat{X}_k$ (see [15]), and by Jensen's inequality, we deduce that $\left\|\hat{X}_k\right\|_2 \leq \left\|\bar{X}_{t_k}\right\|_2$. Recalling (4.15), the inequality (4.21) in Theorem 4.2 gives

$$
\max_{i\in\mathbb{I}_q} \left\|\bar{Y}_{t_k}^i - \hat{Y}_k^i\right\|_1 \quad \leq \quad K(1+\left\|X_0\right\|_2) \sum_{\ell=k}^{m} \left\|\bar{X}_{t_\ell} - \hat{X}_\ell\right\|_2,
$$

for all $k = 0,\dots,m$. In particular, if $X_0 = x_0$ is deterministic, then $\hat{X}_0 = x_0$, and we have an error estimation by quantization of the value function function for the discrete-time optimal switching problem at the initial date measured by:

$$
\max_{i\in\mathbb{I}_q} \left|\bar{v}_i(0,x_0) - \hat{v}_0^i(x_0)\right| \quad \leq \quad K(1+|x_0|) \sum_{k=1}^{m} \left\|\bar{X}_{t_k} - \hat{X}_k\right\|_2 \tag{4.24}
$$

Suppose that one has at hand a global stack of $\bar{N}$ points for the whole space-time grid, to be dispatched with N_k points for each kth-time step, i.e. $\sum_{k=1}^m N_k = \bar{N}$. Then, as in [2], in the case of uniformly elliptic diffusion with bounded Lipschitz coefficients b and σ, one can optimize over the N_k's by using the rate of convergence for the miminal L^2-quantization error given by Zador's theorem:

$$
\left\|\bar{X}_{t_k} - \hat{X}_k\right\|_2 \quad \sim \quad \frac{J_{2,d}\left\|\varphi_k\right\|_{\frac{d}{d+2}}^{\frac{1}{2}}}{N_k^{\frac{1}{d}}} \quad \text{as } N_k \to \infty,
$$

where φ_k is the probability density function of $\bar{X}_{t_k}$, and $\left\|\varphi\right\|_r = (\int |\varphi(u)|^r du)^{\frac{1}{r}}$. From [3], we have the bound $\left\|\varphi_k\right\|_{\frac{d}{d+2}}^{\frac{1}{2}} \leq K\sqrt{t_k}$, for some constant K depending only on b, σ, T, d. Substituting into (4.24) with Zador's theorem, we obtain

$$
\max_{i\in\mathbb{I}_q} \left|\bar{v}_i(0,x_0) - \hat{v}_0^i(x_0)\right| \quad \leq \quad K(1+|x_0|) \sum_{k=1}^{m} \frac{\sqrt{t_k}}{N_k^{\frac{1}{d}}}.
$$

For fixed $h = T/m$ and $\bar{N}$, the sum in the upper bound of the above inequality is minimized over the size of the grids Γ_k, $k = 1,\dots,m$ with

$$
N_k \quad = \quad \left\lceil \frac{t_k^{\frac{d}{2(d+1)}} \bar{N}}{\sum_{k=1}^m t_k^{\frac{d}{2(d+1)}}} \right\rceil,
$$

where $\lceil x \rceil := \min\{k \in \mathbb{N},\ k \geq x\}$, and we have a global rate of convergence given by:

$$\max_{i \in \mathbb{I}_q} \left| \bar{v}_i(0, x_0) - \hat{v}_0^i(x_0) \right| \quad \leq \quad \frac{K(1+|x_0|)}{h(\bar{N}h)^{\frac{1}{d}}}.$$

Actually even with no extra assumptions on b and σ, we have the same estimate, since for all $r > 0$,

$$\left\| \bar{X}_{t_k} - \hat{X}_k \right\|_2 \leq C_{2,r} \left\| \bar{X}_{t_k} \right\|_{2+r} N_k^{-1/d} \leq K N_k^{-1/d},$$

see Lemma 1 in [13].

By combining with the estimate (3.14), we obtain an error bound between the value function of the continuous-time optimal switching problem and its approximation by marginal quantization of order $h^{\frac{1}{2}}$ when choosing a number of points by grid $\bar{N}h$ of order $1/h^{\frac{3d}{2}}$. This has to be compared with the number of points N of lower order $1/h^d$ in the Markovian quantization approach, see Remark 4.1. The complexity of this marginal quantization algorithm is of order $O\left(\sum_{k=1}^{m} N_k N_{k+1}\right)$. In terms of h, if we take $N_k = \bar{N}h = 1/h^{\frac{3d}{2}}$, we then need $O(1/h^{3d+1})$ operations to compute the value function. Recall that the Markovian quantization method requires a complexity of higher order $O(1/h^{4d+1})$, but provides in compensation an approximation of the value function in the whole space grid $\mathbb{X}$.

5 Numerical tests

We test our quantization algorithms by comparison results with explicit formulae for optimal switching problems derived from chapter 5 in [17]. The formulae are obtained for infinite horizon problems, that we adapt to our case by taking as the final gain the (discounted) value function for the infinite horizon problem.

We consider a two-regime switching problem where the diffusion is independent of the regime and follows a geometric Brownian motion, i.e. $b(x,i) = bx$, $\sigma(x,i) = \sigma x$, and the switching costs are constant $c(x,i,j) = c_{ij}$,$i,j = 1,2$. The profit functions are in the form $f_i(t,x) = e^{-\beta t} k_i x^{\gamma_i}$, $i = 1,2$. From Theorem 5.3.5 in [17]), the value functions are given by:

$$\begin{aligned}
v_1(0,x) &= \begin{cases} A_1 x^{m^+} + K_1 k_1 x^{\gamma_1}, & x < \underline{x}_1^* \\ B_2 x^{m^-} + K_2 k_2 x^{\gamma_2} - c_{12}, & x \geq \underline{x}_1^* \end{cases} \\
v_2(0,x) &= \begin{cases} A_2 x^{m^+} + K_2 k_2 x^{\gamma_2}, & x < \underline{x}_2^* \\ A_1 x^{m^+} + K_1 k_1 x^{\gamma_1} - c_{21} & \underline{x}_2^* \leq x \leq \overline{x}_2^* \\ B_2 x^{m^-} + K_2 k_2 x^{\gamma_2}, & x > \overline{x}_2^* \end{cases},
\end{aligned}$$

where A_i, B_i, K_i, $\underline{x}_2^*$ and $\overline{x}_2^*$ depend explicitly on the parameters. In the sequel, we take for value of the parameters:

$$b = 0,\ \sigma = 1,\ c_{01} = c_{10} = 0.5,\ k_1 = 2, k_2 = 1,\ \gamma_1 = 1/3,\ \gamma_2 = 2/3,\ \beta = 1.$$

We compute the value function in regime 2 taken at $X_0 = 3.0$ by means of the first algorithm (Markovian quantization). We take $R = 10X_0$ and vary m, δ and N. The results are compared with the exact value in Table 1. Notice that the algorithm seems to be quite

robust and provides good results even when δm and $\frac{m}{R}$ do not satisfy the constraints given by our theoretical estimates in Remark 4.1.

In Table 2, we have computed the value with the marginal quantization algorithm. We make vary the number of time steps m and the total number of grid points $\bar{N}$ (dispatched between the different time steps as described in Remark 4.3). We have used optimal quantization of the Brownian motion, and the transition probabilities $\pi_k^{ll'}$ were computed by Monte-Carlo simulations with 10^6 sample paths (for an analysis of the error induced by this Monte-Carlo approximation, see Section 4 in [1]). We have also indicated the time spent for these computations. Actually, almost all of this time comes from the Monte-Carlo computations, as the tree descent algorithm is very fast (less than 1s for all the tested parameters).

For the two methods, we look at the impact of the quantization number for each time step (resp. N and $\bar{N}h$) on the precision of the results. As our theoretical estimates showed (see Remarks 4.1 and 4.3), for the first method, increasing N higher than h^{-1} does not seem to improve the precision, whereas for the second method, we can see for several values of h that changing $\bar{N}h$ from h^{-1} to h^{-2} or h^{-3} improves the precision.

Comparing the two tables, the first method seems to provide precise estimates with slightly faster computation times, and it has the further advantage of computing simultaneously the value functions at any points of the space discretization grid $\mathbb{X}$. However, since most of the time spent by our second algorithm was devoted to the calculation of the transition probabilities $\pi_k^{ll'}$, if these were computed beforehand and stored offline, the marginal quantization method becomes more competitive.

$(m, 1/\delta, N)$	$\hat{v}_2(0, 3.0)$	Numerical error (%)	Algorithm time (s)
(10,10,10)	2.1925	3.0	0.2
(10,10,100)	2.1863	2.7	0.5
(10,10,1000)	2.1852	2.7	1.4
(10,100,1000)	2.1882	2.8	8.5
(10,100,5000)	2.1882	2.8	40
(100,10,100)	2.1218	0.31	1.0
(100,10,1000)	2.1213	0.33	8.0
(100,10,5000)	2.1213	0.33	39
(100,100,100)	2.1250	0.16	8.6
(100,100,1000)	2.1250	0.16	82
Exact value	2.1285		

Table 1: Results obtained by Markovian quantization

$(m, \bar{N})$	$\hat{Y}_0^2$	Numerical error (%)	Algorithm time (s)
(10,100)	2.2080	3.7	4.4
(10,1000)	2.2174	4.2	4.9
(10,10000)	2.1276	0.04	5.8
(100,1000)	2.1233	0.24	36
(100,10000)	2.1316	0.15	48
(100,50000)	2.1301	0.07	65
(1000,10000)	2.1161	0.58	353
(1000,50000)	2.1213	0.34	498

Table 2: Results obtained by marginal quantization

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